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Introduction to Sequences and Series

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In this chapter, we will look at:

- ➔ Arithmetic progressions
- ➔ Geometric progressions
- ➔ Infinite sequences
- ➔ Series and sigma notation
- ➔ Arithmetic and Geometric series

Series and sequences are extremely powerful tools. In mathematics, they are for instance related to integrals. Sound engineering is closely related to series. In economy, interests on loans can be calculated using sequences and series.

As always, please free to refer to the book [Croft and Davidson, 2016] for details.

I Introduction

Def. SEQUENCE: A *sequence* is a set of numbers written down in a specific order.

Sequences are very useful for many reasons. The first one is that the elements are likely related by a rule. Studying the sequences hence gives insight on this rule. It also allows to solve puzzles or understand some patterns.

Def. TERM: an element of a sequence is called a *term*

Sometimes we use the symbol $\ldots$ to indicate that the sequence continues.

For example,

$$1, 3, 5, 7, 9$$

and

$$-1, -2, -3, -4, -5$$

are both sequences of five terms.

Also,

$$1, 2, 3, \ldots, 20$$

is the sequence of integers from 1 to 20 inclusive.

All of the sequences given above have a finite number of terms. They are known as finite sequences.

Some sequences go on forever, and these are called *infinite sequences*. To indicate that a sequence might go on forever we can use the $\ldots$ notation.

So when we write the sequence

$$1, 4, 7, 10, 13, 16, 19, 22, 25, \ldots$$

It can be assumed to continue indefinitely.

It is *very* different from

$$1, 4, 7, 10, \ldots, 22, 25$$

which is finite, so be careful.

Main Example

➔ Finite sequence

$$1, -2, 4, -8, 16, -32$$

is a sequence of 6 terms.

Its first term is 1, its second term is -2 and so on.

➔ Infinite sequence

$$1, -2, 4, -8, 16, -32, \ldots$$

if this sequence seems close to the previous one, it is actually an infinite sequence.

Its first term is 1, its second term is -2 and so on.

Exercise 1.

1.1 provide the sequence of the first seven prime numbers

1.2 provide the sequence of the first 10 odd numbers

1.3 provide the sequence of bisextile years between 1900 and 2000 (included)

Very often you will be able to spot a rule that you find the next term in a sequence. For example

$$1, 3, 5, 7, 9, \ldots$$

is a sequence of odd integers and the next term is likely to be 11. The next term in the sequence

$$1, \frac{1}{2}, \frac{1}{4} \ldots$$

might well be $\frac{1}{8}$.

Tip

Be careful, the rule might not be obvious, so do not draw conclusions too quickly !

I a) Notation used for sequences

Often, it is useful to refer to a specific term in a sequence. Consequently, a subscript notation is used to refer to different terms in a sequence.

Def.

DENOTING A TERM: The ith term of a sequence x is referred as x_i

For example, to denote the sequence $1, 4, 7, 10, 13$ by x, the first term can be labeled x_1 , the second term x_2 and so on. That is,

$$x_1 = 1, x_2 = 4, x_3 = 7, x_4 = 10, x_5 = 13$$

and so on if the sequence is longer.

Tip

Sometimes the first term in a sequence is labeled x_0. When it is the case, the second is then labeled x_1, and so.

Main Example

If we have the sequence:

$$1, -2, 4, -8, 16, -32, \ldots$$

we can equivalently say that we have:

$$x_1, x_2, x_3, \ldots$$

with $x_1 = 1, x_2 = -2, x_3 = 4, \ldots$, or, even better, that $x_i = (-2)^{i-1}$.
We could name the sequence with s instead of x:

$$s_1, s_2, s_3, \ldots$$

with $s_i = (-2)^i$.
It is also equivalent to have:

$$x_0, x_1, x_2, x_3, \ldots$$

with $x_0 = 1, x_1 = -2, x_2 = 4, \ldots$, or that $x_i = (-2)^i$. Note that the power, in this case, is "i". So, from time to time, it is more convenient to start with a number that is not 1.

II Arithmetic progressions

Def.

ARITHMETIC PROGRESSION: An arithmetic progression is a sequence where each term is found by adding a fixed quantity, called the *common difference*, to the previous term.
Noting d the common difference, and considering the sequence x, the $i+1$ th term can be deduced from the ith term by

$$x_{i+1} = x_i + d$$

For example, suppose the first term is 1 and we find subsequent terms by repeatedly adding 6. We obtain

$$1, 7, 13, 19, \ldots$$

If we write down the first five terms of the arithmetic progression where the first term is 1 and the common difference is 3, then the second term is 4 (by adding the common difference 3, to the first term). By continuing in this way we can construct the sequence

$$1, 4, 7, 10, 13, \ldots$$

A more general arithmetic progression has first term a and common difference d, that is

- ➔ the first term is a
- ➔ the second term is $a + d$
- ➔ the third term is $a + 2d$
- ➔ the fourth term is $a + 3d$
- ➔ and so on.

This leads to the following formula for the nth term.

Prop.

NTH TERM OF AN ARITHMETIC PROGRESSION: The nth term of an arithmetic progression denoted by x, initialized with x_1 as the first term and d as the common difference, is given by:

$$x_n = x_1 + (n-1)d$$

Let's use a proof by induction:

- Base case (initialization)
  The first term is x_1, which means that $x_2 = x_1 + d$. Or, having $n = 2$, we have $x_2 = x_1 + (2-1) \times d$, which correspond to the formula. The formula is correct for the initialization.

- The step case
  Let's assume the formula is true for the rank $n-1$. so we have $x_{n-1} = x_1 + \big((n-1)-1\big)d$. Or, we have

  $$x_n = x_{n-1} + d$$

  which means that, replacing x_{n-1}:

  $$x_n = x_1 + (n-2)d + d$$

  and hence

  $$x_n = x_1 + (n-1)d$$

  The formula holds

These two steps, using induction, prove that the formula is correct.
Et voilà !

Main Example

- We can use the formula to find the 10th term of an arithmetic progression with first term 3 and common difference 5.

  $$a = 3, d = 5, \text{ and } n = 10$$

  The 10th term is calculated, noting s the sequence, as:

  $$s_{10} = 3 + (101) \times 5 = 3 + 9 \times 5 = 48$$

  The 10th term is $s_{10} = 48$.

- If Pauline is saving up 100$ per month, how much will she have after 5 years ?
  We have a sequence with the first term being 100 (she starts the 1st month with 100$, or we have to start at month 0), and the common difference is 100. 5 years means 60 months, hence $n = 60$.

  She will have saved

  $$\begin{aligned} s_{60} &= s_1 + (n-1)d \\ &= 100 + (60-1) * 100 \\ &= 6000 \end{aligned}$$

Exercise 2.
Find the nth term (in the sense x_n is the nth term of a sequence) of the arithmetic progression when:

2.1 $x_0 = 10, d = -3, n = 5$

2.2 $x_1 = 10, d = -3, n = 5$

2.3 $s_0 = -20, d = 9, n = 12$

2.4 $s_3 = -4, d = 12, n = 8$

2.5 $v_1 = 2, d = -\frac{1}{2}, n = 10$

2.6 $x_1 = 1, d = -2.4, n = 11$

2.7 $x_1 = 0, d = 5, n = 3$

Exercise 3.

3.1 Does the number 203 belongs to the arithmetic progression with $x_1 = 3$ and $d = 4$?

3.2 Does the number 12 belongs to the arithmetic progression with $x_1 = 210$ and $d = -13$?

Exercise 4.

If Pauline is saving up 100$ per month, how long will it take for her to save 10000$?

III Geometric progressions

Def.

GEOMETRIC PROGRESSION: A geometric progression is a sequence where each term is found by multiplying the previous term by a fixed quantity called the *common ratio*. Noting r the common ratio and considering the sequence x, the $i+1$ term can be deduced from the ith term by

$$x_{i+1} = r x_i$$

For example, suppose the first term is 2 and we find subsequent terms by repeatedly multiplying by 5. We obtain the sequence

$$2, 10, 50, 250, \ldots$$

If we write down the geometric progression whose first term is 1 and whose common ratio is 1/2, we obtain the second term by multiplying the first by the common ratio, 1/2, that is

$$1/2x1 = 1/2$$

Continuing in this way we obtain the sequence as:

$$1, 1/2, 1/4, 1/8, \ldots$$

A general geometric progression has first term a and common ratio r, and can therefore be written as

- ➔ the first term is a
- ➔ the second term is ar
- ➔ the third term is ar^2
- ➔ the fourth term is ar^3
- ➔ and so on.

This leads to the following formula for the nth term.

Prop.

*n*TH TERM OF AN GEOMETRIC PROGRESSION: The nth term of an geometric progression denoted by x, initialized with x_1 as the first term and r as the common ratio, is given by:

$$x_n = x_1 r^{n-1}$$

Let's use a proof by induction:

- Base case (initialization)
  The first term is x_1, which means that $x_2 = x_1 \times r$. Or, having $n = 2$, we have $x_2 = x_1 \times r^{2-1}$, which correspond to the formula. The formula is correct for the initialization.

- The step case
  Let's assume the formula is true for the rank $n-1$. so we have $x_{n-1} = x_1 r^{n-1-1}$. Or, we have

  $$x_n = x_{n-1} \times r$$

  which means that, replacing x_{n-1}:

  $$x_n = x_1 r^{n-2} \times r$$

  and hence

  $$x_n = x_1 r^{n-1}$$

  The formula holds

These two steps, using induction, prove that the formula is correct.
Et voilà !

Main Example

- Let's use the formula to find the 7th term of a geometric progression with first term 2 and common ratio 3.

  $$s_1 = 2, r = 3, \text{ and } n = 7$$

  7th term is

  $$s_7 = 2 \times 3^{71} = 2 \times 3^6 = 1458$$

  therefore the 7th term is 1458.

- If Pauline has saved up 100$ on her saving accounts, and if her saving accounts has a saving rates of 2.3%, how much will she have after 5 years ? After 10 years?
  We have a sequence with the first term being 100. It is a saving account, so she get an extra 2.3%, i.e., her money is multiplied by 1.023 each year. Consequently, the common ratio is 1.023

  - 5 years means $n = 5$.
    She will have saved

    $$\begin{aligned} s_5 &= s_1 r^{n-1} \\ &= 100 \times 1.023^4 \\ &= 109.52 \end{aligned}$$

  - 10 years means $n = 10$.
    She will have saved

    $$\begin{aligned} s_{10} &= s_1 r^{n-1} \\ &= 100 \times 1.023^9 \\ &= 122.71 \end{aligned}$$

Exercise 5.
Find the nth term (in the sense x_n is the nth term of a sequence) of the geometric progression when:

5.1 $x_0 = 10, r = -2, n = 5$

5.2 $x_1 = 10, r = -3, n = 5$

5.3 $s_0 = -20, r = 0.9, n = 12$

5.4 $s_3 = -4, r = 1.2, n = 8$

5.5 $v_1 = 2, r = -\frac{1}{2}, n = 10$

5.6 $x_1 = 1, r = -0.4, n = 11$

5.7 $x_1 = 0, r = 15, n = 3$

Exercise 6.

6.1 Does the number 48 belongs to the geometric progression with $x_1 = 3072$ and $r = 0.5$?

6.2 Does the number 6072 belongs to the geometric progression with $x_1 = 3$ and $4 = -2$?

Exercise 7.
If Pauline has saved up 1000$, and have a saving rates of 4.3%, how long will it take for her to save 10000$?

IV Infinite sequences

Def.

INFINITE SEQUENCE: A sequences that continues indefinitely is called an *infinite sequence.*

Recall that we can use the ... notation to indicate this, thus

$$1, 1/2, 1/3, 1/4, 1/5, \ldots$$

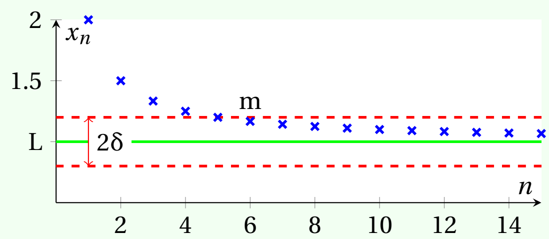
is the infinite sequence of inverse of positive whole numbers. The sequence can be written in the abbreviation form

$$x_k = 1/k, \text{for } k = 1, 2, 3, \ldots$$

As k gets larger and larger and approaches infinity, the term of the sequence get closer and closer to zero. This gives us the definition of a limit:

Def.

LIMIT: L is the limit of a sequence x if for any distance δ, we can find a m large enough so all the x_n that follows x_m are close to L up to δ.



It means that, when n goes to infinity, x_n goes to L.
We write this concisely as:

$$\lim_{n \to \infty} x_n = L$$

In the previous example, we have:

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

lim is an abbreviation for limit, so $\lim_{k\to\infty}$ means we must examine the behaviour of the sequence as k gets larger and larger. When a sequence possesses a limit it is said to *converge*. However, not all sequences possess a limit. The sequence defined by

$$x_k = 3k - 2$$

which is

$$1, 4, 7, 10 \ldots$$

is one such example. As k gets larger and larger so too do the terms of the sequence, this sequence is said to *diverge*.

Let's illustrate this on the sequence

$$x_k = 3 + 1/k^2, k = 1, 2, 3, \ldots$$

When k gets larger, $1/k^2$ tends to zero, and hence

$$\lim_{k\to\infty} 3 + \frac{1}{k^2} = 3$$

The sequence x_k converges to the limit 3.

V Series and sigma notation

Whenever the terms of a sequence are added together, we obtain what is know as a *series*.

Def.

SERIES: An series is a sequence where the nth term is formed with the sum of the first n terms of a given sequence. If the series is noted S and the sequence x:

$$S_n = x_1 + x_2 + \ldots + x_n$$

For example, if we have the sequence $1, 1/2, 1/4, 1/8$, we obtain the series S, where

- ➔ $S_1 = 1$
- ➔ $S_2 = 1 + \frac{1}{2}$
- ➔ $S_3 = 1 + \frac{1}{2} + \frac{1}{4}$
- ➔ $S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

This series ends after the 4th term and is said to be a finite series. Other series we shall meet continue indefinitely and are said to be *infinite series*.

V a) Sigma notation

Sigma notation, $\sum$, provides a concise and convenient way of writing long sums. The sum

$$1+2+3+\ldots+10+11+12$$

can be written very concisely using the capital Greek letter $\sum$ as

$$\sum_{k=1}^{k=12} k$$

Def.

SIGMA NOTATION $\sum$: The sigma notation, or the sum notation, appears as the symbol, $\sum$, which is the Greek upper case letter, S. The summation sign, $\sum$, instructs us to sum the elements of a sequence. A typical element of the sequence which is being summed appears to the right of the summation sign. The variable of summation, i.e. the variable which is being summed.

stopping point

summation sign $\quad \sum_{k=1}^{n} x_k \quad$ element to be summed

index of summation

starting point

The $\sum$ stands for for a sum, in this case the sum of all the value of k as k ranges through all whole numbers from 1 to 12. Note that the lowermost and uppermost values of k are written at the bottom and top of the sigma sign respectively. The lowermost value of k is commonly $k = 1$ or $k = 0$, but other values are certainly possible. Sometimes the sigma notation itself is abbreviated. The $k =$ part, written at the bottom and the top of the sigma sign, can be omitted (if there is no possible confusion !) and k could be represented by other letters like n or i. k is known as a *dummy variable.*

For example, the following expressions are identical:

$$\sum_{k=1}^{k=12} k = \sum_{i=1}^{i=12} i = \sum_{k=1}^{12} k = \sum_{1}^{12}$$

Main Example

For instance, if we want to write out explicitly what is meant by

$$\sum_{k=1}^{k=5} k^3$$

k ranges from 1 to 5, and if we cube each value of k, it leads to

$$\sum_{k=1}^{k=5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + r^3$$

If we want to write out explicitly what is meant by

$$\sum_{k=1}^{k=4} (-1)^k 2^k$$

- ➔ when $k = 1$, $(-1)^k 2^k = (-1)^1 2^1 = -2$
- ➔ when $k = 2$, $(-1)^k 2^k = (-1)^2 2^2 = 4$
- ➔ when $k = 3$, $(-1)^k 2^k = (-1)^3 2^3 = -8$
- ➔ when $k = 4$, $(-1)^k 2^k = (-1)^4 2^4 = 16$

So

$$\sum_{k=1}^{k=4} (-1)^k 2^k = -2 + 4 - 8 + 16$$

Exercise 8.

8.1 What is $S = \sum_{k=1}^{4} k$?

8.2 What is $S = \sum_{k=4}^{9} k$?

8.3 What is $S = \sum_{k=1}^{4} k^3$?

8.4 What is $S = \sum_{k=1}^{5} 2k$?

8.5 What is $S = \sum_{k=1}^{n} 1$?

V b) Arithmetic series

Def.

ARITHMETIC SERIES: An arithmetic series is a series based on an arithmetic progression. In other words, an arithmetic series is a sequence where the nth term is formed with the sum of the first n terms of an arithmetic progression. If we note S the arithmetic series and x the arithmetic progression:

$$S_n = \sum_{k=1}^{n} x_k$$

For example, the arithmetic progression with five terms having first term 4 and common difference 5 is

$$4, 9, 14, 19, 24$$

If these terms are added we obtain the arithmetic series

$$S_5 = 4 + 9 + 14 + 19 + 24$$

It is easily verified that this has sum 70. But when the given series has a larger number of terms then finding its sum by directly adding all the terms will be laborious. Fortunately there is a formula that enables us to find the sum of an arithmetic series.

Prop.

ARITHMETIC SERIES: If S is the arithmetic series formed from the arithmetic progression of first term a and common difference d, then:

$$S_n = \frac{n\big(2a + (n-1)d\big)}{2}$$

Let's write x the arithmetic progression, with $x_1 = a$:

$$x_n = a + (n-1)d$$

Let's use a proof by induction:

- ➔ Base case (initialization)
  The first term is S_1 is the sum of the first terms, i.e.

  $$S_1 = x_1$$

  Or, having $n = 1$, we have $S_1 = \frac{1\big(2a+(1-1)d\big)}{2} = a$, which correspond to the formula. The formula is correct for the initialization.

- ➔ The step case
  Let's assume the formula is true for the rank $n-1$. so we have $S_{n-1} = \dfrac{(n-1)\times\big(2a+(n-1-1)d\big)}{2}$. Or, we have

  $$x_n = a + (n-1)d$$

  and, $S_n = \sum\limits_{k=1}^{n} x_k = \sum\limits_{k=1}^{n-1} x_k + x_n = S_{n-1} + x_n$ So

  $$\begin{aligned} S_n &= \frac{(n-1)\times\big(2a+(n-2)d\big)}{2} + a + (n-1)d \\ &= \frac{(n-1)\times\big(2a+(n-2)d\big) + 2\times\big(a+(n-1)d\big)}{2} \\ &= \frac{2(n-1+1)a + (n-1)(n-2+2)d}{2} \\ &= \frac{2na + n(n-1)d}{2} \\ &= \frac{n\big(2a+(n-1)d\big)}{2} \end{aligned}$$

  The formula holds

These two steps, using induction, prove that the formula is correct.
Et voilà !

Main Example

➔ For finding the first 10 terms of the arithmetic series with first term 3 and common difference 4.

We use the formula $S_n = \dfrac{n\big(2a+(n-1)d\big)}{2}$. with $n = 10, a = 3$, and $d = 4$:

$$\begin{aligned} S_{10} &= \frac{10\times\big(2\times 3+(10-1)\times 4\big)}{2} \\ &= 5(6+36) \\ &= 210 \end{aligned}$$

➔ The sum of the first 15 terms of an arithmetic series is 165. The common difference is 2. Can we calculate the first term of the series ?

We use the formula $S_n = \dfrac{n\big(2a+(n-1)d\big)}{2}$, with $n = 15, S_n = 165, d = 2$. To find a:

$$\begin{aligned} S_n = \frac{n\big(2a+(n-1)d\big)}{2} &\Leftrightarrow 2S_n = n\big(2a+(n-1)d\big) \\ &\Leftrightarrow \frac{2S_n}{n} = 2a+(n-1)d \\ &\Leftrightarrow \frac{2S_n}{n} - (n-1)d = 2a \\ &\Leftrightarrow a = \frac{S_n}{n} - \frac{(n-1)d}{2} \end{aligned}$$

And hence:

$$a = \frac{165}{15} - \frac{(15-1)\times 2}{2} = -3$$

Exercise 9.

9.1 What are the five first terms of the arithmetic series with $x_1 = 10, d = 2$?

9.2 What are the seven first terms of the arithmetic series with $x_1 = 10, d = -2$?

9.3 What are the five six terms of the arithmetic series with $x_1 = 100, d = -45$?

V c) Geometric series

Def.

GEOMETRIC SERIES: An geometric series is a series based on an geometric progression. In other words, a geometric series is a sequence where the nth term is formed with the sum of the first n terms of an geometric progression. If we note S the geometric series and x the geometric progression:

$$S_n = \sum_{k=1}^{n} x_k$$

For example, the geometric progression with five terms having first term 2 and common ratio 3 is $2, 6, 18, 54, 162$.

If these terms are added we obtain the geometric series $2+6+18+54+162$. It is easily verified that this has sum 242. Again similar to arithmetic series to find the sum of the geometric series for a larger number of terms there is a formula.

Prop.

GEOMETRIC SERIES:

$$S_n = \frac{a(1-r^n)}{1-r}$$

provided $r \neq 1$.

Let's write x the geometric progression, with $x_1 = a$ and the common ratio r:

$$x_n = ar^{n-1}$$

Let's use a proof by induction:

➔ Base case (initialization)
The first term is S_1 is the sum of the first terms, i.e.

$$S_1 = x_1$$

Or, having $n = 1$, we have $S_1 = a\frac{1-r^1}{1-r} = a$, which correspond to the formula. The formula is correct for the initialization.

➔ The step case
Let's assume the formula is true for the rank $n-1$. so we have $S_{n-1} = a\dfrac{1-rn-1}{1-r}$. Or, we have

$$x_n = ar^{n-1}$$

and, $S_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n-1} x_k + x_n = S_{n-1} + x_n$ So

$$\begin{aligned} S_n &= a\frac{1-r^{n-1}}{1-r} + ar^{n-1} \\ &= a\frac{1-r^{n-1} + r^{n-1} \times (1-r)}{1-r} \\ &= a\frac{1-\cancel{r^{n-1}} + \cancel{r^{n-1}} - r^n}{1-r} \\ &= a\frac{1-r^n}{1-r} \end{aligned}$$

The formula holds

These two steps, using induction, prove that the formula is correct.
Et voilà !

The formula excludes the use of $r = 1$ because in this case the denominator becomes zero. If it is the case, the sequence should rather been considered as an arithmetic series with $d = 0$.

Main Example

Let's find the sum of the first terms of the geometric series with first term 2 and common ratio 3.
Using the formula

$$S_n = \frac{a(1-r^n)}{1-r}$$

with $n = 5, a = 2, r = 3$:

$$S_5 = \frac{2(1-e^5)}{1-3} = 242$$

Going further

Geometric series are pivotal in the study of fractals. Fractals are very common in nature ! For instance, snowflakes, frost, sponges and waves can be described as fractals.

wikipedia

V d) Infinite geometric series

When the terms of an infinity sequence are added we obtain an infinite series. It may seem strange to try to add together an infinite number of terms but under some circumstances their sum is finite and can be found. The sum of an infinite number of terms of a geometric series is denoted by

$$S_\infty = \frac{a}{1-r}, -1 < r < 1$$

Note that the common ratio should lies between -1 and 1.

Tip

The ratio has to be between -1 and 1 ! In this case, the formula for the sum can be taken to the limit:

$$\lim_{n\Rightarrow\infty} S_n = \lim_{n\Rightarrow\infty} \frac{a(1-r^n)}{1-r}$$

Because $|r| < 1$, $\lim_{n\Rightarrow\infty} r^n = 0$!

Main Example

Let's find the sum of the infinite geometric series with first term 2 and common ratio $\frac{1}{3}$.
Using the formula $S_\infty = \frac{a}{1-r}, -1 < r < 1$, with $a = 2$ and $r = \frac{1}{3}$:

$$\begin{aligned} S_\infty &= \frac{2}{1-\frac{1}{3}} \\ &= \frac{2}{\frac{2}{3}} \\ &= \frac{2\times 3}{2} \\ &= 3 \end{aligned}$$

Note that we can only make use of the formula because the value of r lies between -1 and 1.

Geometric series can do other cool stuff !
For instance, we can use them to convert the number m:

$$m = 1.2121212121\ldots$$

to a fraction ! m is actually:

$$m = 1 + 0.21 + 0.0021 + 0.000021 + \ldots$$

which is:

$$m = 1 + \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \dots$$

or,

$$m = 1 + \frac{21}{100} + \frac{21}{100} \times \left(\frac{1}{100}\right) + \frac{21}{100} \times \left(\frac{1}{100}\right)^2 + \dots$$

and finally, we have:

$$m = 1 + \sum_{k=0}^{\infty} \frac{21}{100} \left(\frac{1}{100}\right)^k$$

It is the limit of the geometric series of the ratio $r = \frac{1}{100}$ and first term $\frac{21}{100}$. So now we can apply the formula for geometric series:

$$m = 1 + \frac{21}{100} \frac{1}{1 - \frac{1}{100}} = 1 + \frac{21}{100 - 1} = 1 + \frac{21}{99}$$

VI Solutions to exercises

Solution 1.

1.1 $1, 2, 3, 5, 7, 11, 13$

1.2 $1, 3, 5, 7, 9, 11, 13, 15, 17, 19$

1.3 $1904, 1908, 1912, 1916, 1920, 1924, 1928, 1932, 1936, 1940, 1944, 1948, 1952, 1956, 1960, 1964, 1968, 1972, 1976, 1980, 1984, 1988, 1992, 1996, 2000$

Solution 2.

2.1 $x_0 = 10$, with $d = -3$ means $x_1 = 7$. $x_5 = 7 + (5-1) \times (-3) = -5$

2.2 $x_5 = 10 + (5-1) \times (-3) = -2$

2.3 $s_0 = -20, d = 9, n = 12$ means $s_1 = -20 + 9 = -11$. $s_{12} = -11 + (12-1) \times 9 = 88$

2.4 $s_3 = -4, d = 12, n = 8$ means $s_3 = s_1 + (3-1) \times 12$ and hence $s_1 = s_3 - (3-1) \times 12 = -28$. $s_8 = -28 + (8-1) \times 12 = 56$.

2.5 $v_{10} = 2 + (10-1) \times \frac{1}{2} = 6.5$

2.6 $x_{11} = 1 + (11-1) \times (-2.4) = -23$

2.7 $x_3 = 0 + (3-1) * 5 = 10$

Solution 3.

3.1 $x_n = x_1 + (n-1)d$ and hence $x_n = -1 + 4n$. Let's solve $4n - 1 = 203$, which leads to $4n = 204$ and $n = 51$. It is an integer, hence 203 belongs to the sequence (and $x_{51} = 203$).

3.2 $x_n = x_1 + (n-1)d$ and hence $x_n = 223 - 13n$. Let's solve $223 - 13n = 12$, which leads to $13n = 211$ and $n = 16.23$. It is not an integer, hence 12 does not belong to the sequence.

Solution 4.
We have a sequence with the first term being 100 (she starts the 1st month with 100$, or we have to start at month 0), and the common difference is 100. After n months, she will have saved $s_n = s_1 + (n-1)d$.

$$\begin{aligned} s_n = s_1 + (n-1)d \quad &\Leftrightarrow \quad s_n - s_1 = (n-1)d \\ &\Leftrightarrow \quad \frac{s_n - s_1}{d} = n-1 \\ &\Leftrightarrow \quad n = 1 + \frac{s_n - s_1}{d} \\ &\Leftrightarrow \quad n = 1 + \frac{10000-100}{100} \\ &\Leftrightarrow \quad n = 100 \end{aligned}$$

So she needs 100 months, i.e., slightly more than 8 years.

Solution 5.

5.1 $x_0 = 10$, with $r = -3$ means $x_1 = -30$. $x_5 = x_1 \dfrac{1-r^{n-1}}{1-r} = -2430$

5.2 $x_5 = 810$

5.3 $s_0 = -20, r = 0.9, n = 12$ means $s_1 = -20 \times 9 = -180$. $s_{12} = 56.5$

5.4 $s_3 = -4, r = 1.2, n = 8$ means $s_3 = s_1 r^2$ and hence $s_1 = s_3 / r^2 = -2.78$. $s_8 = -9.96$.

5.5 $v_{10} = 2 + (10-1) \times \dfrac{1}{2} = -0.0020$

5.6 $x_{11} = 0.0001$

5.7 $x_3 = 0$

Solution 6.

6.1 $x_n = x_1 r^{n-1}$. Let's solve $48 = 3072 \times (1/2)^n$, which leads to $\log \dfrac{48}{3072} = n \log \dfrac{1}{2}$ and $n = 6$. It is an integer, hence 48 belongs to the sequence (and $x_6 = 48$).

6.2 $x_n = x_1 r^{n-1}$. Let's solve $6072 = 3 \times (-2)^n$, which, squared, leads to $\log \dfrac{6072^2}{3^2} = 2n \log 2$ and $n = 10.98$. It is not an integer, hence 6072 does not belong to the sequence.

Solution 7.
We have a sequence with the first term being 1000, and the common ratio of 1.043. After n years, she will have saved $s_n = s_1 r^{n-1}$.

$$\begin{aligned} s_n = s_1 r^{n-1} \quad &\Leftrightarrow \quad \frac{s_n}{s_1} = r^{n-1} \\ &\Leftrightarrow \quad \log\left(\frac{s_n}{s_1}\right) = \log\left(r^{n-1}\right) \\ &\Leftrightarrow \quad \log\left(\frac{s_n}{s_1}\right) = (n-1)\log(r) \\ &\Leftrightarrow \quad \frac{\log\left(\frac{s_n}{s_1}\right)}{\log(r)} = n-1 \\ &\Leftrightarrow \quad n = 1 + \frac{\log\left(\frac{s_n}{s_1}\right)}{\log(r)} \\ &\Leftrightarrow \quad n = 1 + \frac{\log\left(\frac{10000}{1000}\right)}{\log(1.043)} \\ &\Leftrightarrow \quad n = 56 \end{aligned}$$

So she needs 56 years.

Solution 8.

8.1 $S = 1+2+3+4 = 10$

8.2 $S = 39$

8.3 $S = 1^3 + 2^3 + 3^3 + 4^3 = 100$

8.4 $S = 30$

8.5 there is n terms in the sum, each being 1, hence $S = n$

Solution 9.

9.1 $10, 22, 36, 52, 70$

9.2 $10, 18, 24, 28, 30, 30, 28$

9.3 $100, 155, 165, 130, 50, -75$

Bibliography

[Croft and Davidson, 2016] Croft, A. and Davidson, R. (2016). *Foundation Maths.* Pearson.