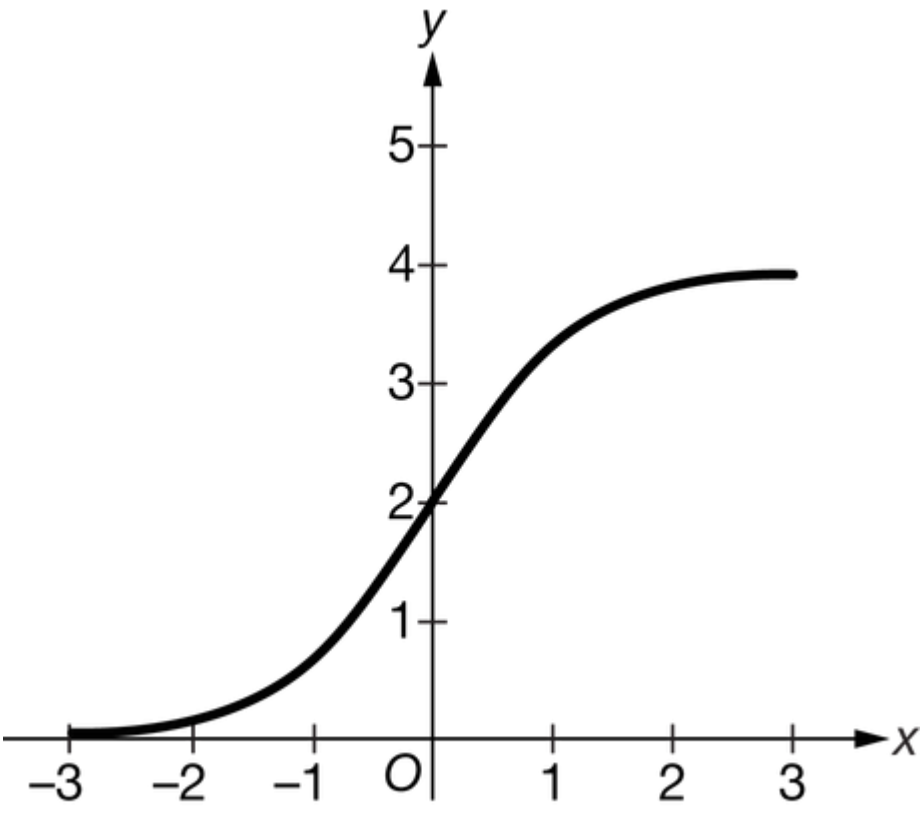
Unit 5 Progress Check: FRQ Part B

1. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.



Graph of g

The graph of the continuous function g is shown above for $-3 \leq x \leq 3$. The function g is twice

Unit 5 Progress Check: FRQ Part B

differentiable, except at $x = 0$.

Let f be the function with $f(0) = -1$ and derivative given by $f'(x) = \left(x^2 - \frac{5}{4}\right)e^x$.

(a) Find the x-coordinate of each critical point of f. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

Please respond on separate paper, following directions from your teacher.

(b) Find all values of x at which the graph of f has a point of inflection. Give reasons for your answers.

Please respond on separate paper, following directions from your teacher.

(c) Fill in the missing entries in the table below to describe the behavior of g' and g'' on the interval $-3 \leq x \leq 3$. Indicate Positive or Negative. Give reasons for your answers.

x	$x = -3$	$-3 < x < 0$	$x = 0$	$0 < x < 3$	$x = 3$
$g(x)$	0	Positive	2	Positive	4
$g'(x)$	0		2		0
$g''(x)$	0		Undefined		0

Please respond on separate paper, following directions from your teacher.

(d) Let h be the function defined by $h(x) = f(x)\,g(x)$. Is h increasing or decreasing at $x = 0$? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

Part A

Unit 5 Progress Check: FRQ Part B

Note: Sign charts are a useful tool to investigate and summarize the behavior of a function. By itself a sign chart for $f'(x)$ or $f''(x)$ is not a sufficient response for a justification.

The first point requires reference to $f'(x) = 0$.

A maximum of 1 out of 3 points is earned for only one correct critical point with correct identification and justification, and no incorrect critical points are included.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3 ✔

The student response accurately includes all three of the criteria below.

- ☐ critical points
- ☐ relative maximum at $x = -\frac{\sqrt{5}}{2}$ with justification
- ☐ relative minimum at $x = \frac{\sqrt{5}}{2}$ with justification

Solution:

$f'(x) = \left(x^2 - \frac{5}{4}\right)e^x = 0 \Rightarrow x^2 = \frac{5}{4} \Rightarrow x = -\frac{\sqrt{5}}{2},\ x = \frac{\sqrt{5}}{2}$

f has a relative maximum at $x = -\frac{\sqrt{5}}{2}$ because $f'(x)$ changes from positive to negative there.

f has a relative minimum at $x = \frac{\sqrt{5}}{2}$ because $f'(x)$ changes from negative to positive there.

Part B

Note: Sign charts are a useful tool to investigate and summarize the behavior of a function. By itself a sign chart for $f'(x)$ or $f''(x)$ is not a sufficient response for a justification.

A maximum of 1 out of 2 points is earned if only one point of inflection with reason and no incorrect points of inflection.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

Unit 5 Progress Check: FRQ Part B

0	1	✔ 2

The student response accurately includes both of the criteria below.

- ☐ $f''(x)$
- ☐ answers with reasons

Solution:

$$f''(x) = 2xe^x + \left(x^2 - \tfrac{5}{4}\right)e^x = e^x\left(x^2 + 2x - \tfrac{5}{4}\right) = 0$$
$$\Rightarrow\ x^2 + 2x - \tfrac{5}{4} = 0$$
$$\Rightarrow\ x = \frac{-2 \pm \sqrt{4 - 4(1)\left(-\frac{5}{4}\right)}}{2} = \frac{-2 \pm \sqrt{9}}{2} = \frac{-2 \pm 3}{2}$$
$$\Rightarrow\ x = -\tfrac{5}{2} \text{ and } x = \tfrac{1}{2}$$

Because $f''(x)$ changes sign at $x = -\frac{5}{2}$ and $x = \frac{1}{2}$, the graph of f has points of inflection there.

Part C

A maximum of 1 out of 2 points is earned if only behavior for $g'(x)$ with reason OR behavior for $g''(x)$ with reason.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	✔ 2

The student response accurately includes both of the criteria below.

- ☐ behavior for $g'(x)$ and $g''(x)$
- ☐ reasons

Unit 5 Progress Check: FRQ Part B

Solution:

x	$x=-3$	$-3<x<0$	$x=0$	$0<x<3$	$x=3$
$g(x)$	0	Positive	2	Positive	4
$g'(x)$	0	**Positive**	2	**Positive**	0
$g''(x)$	0	**Positive**	Undefined	**Negative**	0

$g'(x)$ is positive for $-3 < x < 0$ and $0 < x < 3$ because g is increasing there.

$g''(x)$ is positive for $-3 < x < 0$ because the graph of g is concave up there, and $g''(x)$ is negative for $0 < x < 3$ because the graph of g is concave down there.

Part D

The response is not eligible for the second point with an incorrect product rule.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2 ✔

The student response accurately includes both of the criteria below.

- ☐ product rule
- ☐ answer with reason

Solution:

$h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

$h'(0) = f'(0)\,g(0) + f(0)\,g'(0)$

Because $f(0) < 0,\ f'(0) < 0,\ g(0) > 0$, and $g'(0) > 0$, it follows that $h'(0) < 0$ and h is decreasing at $x = 0$.

Unit 5 Progress Check: FRQ Part B

2. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

The number of mosquitoes in a field after a major rainfall is modeled by the function M defined by $M(t) = -t^3 + 12t^2 + 144t$, where t is the number of days after the rainfall ended and $0 \le t \le 18$.

(a) Using correct units, interpret the meaning of $M'(15) = -171$ in the context of the problem.

Please respond on separate paper, following directions from your teacher.

(b) Based on the model, what is the absolute maximum number of mosquitoes in the field over the time interval $0 \le t \le 18$? Justify your answer.

Please respond on separate paper, following directions from your teacher.

(c) For what values of t is the rate of change of the number of mosquitoes in the field increasing?

Unit 5 Progress Check: FRQ Part B

Please respond on separate paper, following directions from your teacher.

(d) For $0 \leq t \leq 18$, the number of bats in the field is modeled by the differentiable function B, where B is a function of the number of mosquitoes in the field. Based on the models, write an expression for the rate of change of the number of bats in the field at time $t = a$.

Please respond on separate paper, following directions from your teacher.

Part A

The point requires a time reference, rate, and units.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1 ✓

The student response accurately includes a correct interpretation of $M'(15) = 171$

Solution:

At time $t = 15$ days, the number of mosquitoes in the field is decreasing at a rate of 171 mosquitoes per day.

Part B

Note: Sign charts are a useful tool to investigate and summarize the behavior of a function. By itself a sign chart for $f'(x)$ or $f''(x)$ is not a sufficient response for a justification.

The response is eligible for additional points based on consistent answers using an incorrect $M'(t)$ with a maximum of one computational error.

The fourth point does not require a simplified answer.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

Unit 5 Progress Check: FRQ Part B

0	1	2	3	4 ✔

The student response accurately includes all four of the criteria below.

- ☐ $M'(t)$
- ☐ sets $M'(t) = 0$
- ☐ identifies $t = 12$ as a candidate
- ☐ answer with justification

Solution:

$M'(t) = -3t^2 + 24t + 144 = -3\left(t^2 - 8t - 48\right) = -3(t - 12)(t + 4)$

$M'(t) = 0 \Rightarrow t = 12$

Because $M'(t)$ changes from positive to negative at $t = 12$, M has a relative maximum at $t = 12$.

Because $t = 12$ is the only critical point of M on the interval $0 \le t \le 18$, that is the location of a relative maximum. It is also the location of the absolute maximum of M on the interval $0 \le t \le 18$.

$M(12) = -12^3 + 12\left(12^2\right) + 144(12) = 144(12) = 1728$

The absolute maximum number of mosquitoes in the field over the time interval $0 \le t \le 18$ is $M(12) = 1728$.

Part C

The third point may be earned based on a consistent answer using an incorrect $M''(t)$ with a maximum of one computational error.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3 ✔

Unit 5 Progress Check: FRQ Part B

The student response accurately includes all three of the criteria below.

- ☐ considers $M''(t)$
- ☐ expression for $M''(t)$
- ☐ answer

Solution:

$M''(t) = -6t + 24 = -6(t - 4)$

$M''(t) = -6(t - 4) > 0 \Rightarrow t < 4$

The rate of change of the number of mosquitoes in the field is increasing for $0 \leq t \leq 4$.

Part D

The point is earned with $B'(M(a)) \cdot M'(a)$

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1 ✔

The student response accurately includes a correct expression.

Solution:

The number of bats in the field at time $t = a$ is given by $B(M(a))$.

Therefore, the rate of change of the number of bats in the field at time $t = a$ is given by $B'(M(a)) \cdot M'(a) = B'\left(-a^3 + 12a^2 + 144a\right) \cdot \left(-3a^2 + 24a + 144\right)$.