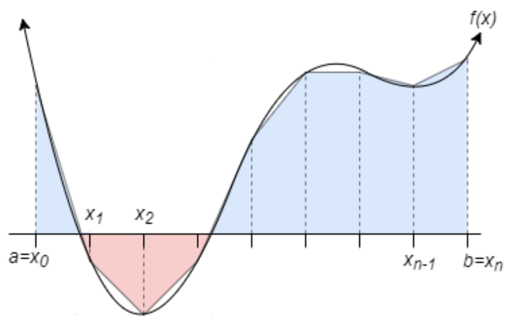
5.5 - Integral Approximation Techniques

Mr. Guillen's AP Calculus

We used rectangles in section 5.1 to approximate the area under a curve, but we can try to get a closer value by approximating the function as a series of connected lines or parabolas. Trapezoid rule replaces the function using straight lines, whereas Simpson's rule replaces the function with quadratics.

Trapezoid Rule

The integral of f on the interval $[a, b]$ can be approximated using n trapezoids of equal width $\Delta x = \frac{b-a}{n}$.



To derive the trapezoid rule, we just have to sum the area of n trapezoids, for which in this case the area of a trapezoid is $b \cdot \text{avg}(h)$, where b is the length of the base and $\text{avg}(h)$ is the average height. Since each trapezoid has a base of Δx, the total area of the trapezoids will be

$$\begin{aligned}\int_a^b f(x)dx &\approx \Delta x \cdot \frac{f(x_0)+f(x_1)}{2} + \Delta x \cdot \frac{f(x_1)+f(x_2)}{2} + \ldots + \Delta x \cdot \frac{f(x_{n-1})+f(x_n)}{2} \\ &= \frac{\Delta x}{2} \cdot (f(x_0) + f(x_1) + f(x_1) + \ldots + f(x_{n-1}) + f(x_{n-1}) + f(x_n)) \\ &= \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n))\end{aligned}$$

Example 1

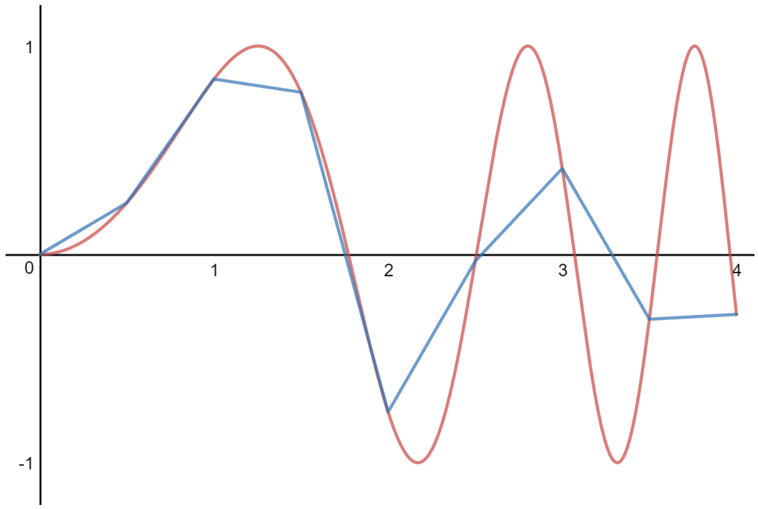
Approximate the area under $f(x) = \sin(x^2)$ on the interval $[0, 4]$ using trapezoid rule and 8 trapezoids.

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{\Delta x}{2} \cdot (f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)) \\ &= \frac{1}{4} \cdot (\sin(0) + 2\sin(0.25) + 2\sin(1) + \cdots + 2\sin(12.25) + \sin(16)) \\ &= 0.517007...\end{aligned}$$

If we graph our approximation to this function, we see that it does a good job at approximating the function on the interval $[0, 2]$, but as the function becomes more 'curvy' the trapezoid rule does a poor job. In general, the error of the trapezoid rule can be bounded by

$$|E_T| \leq \frac{b-a}{12}\Delta x^2 k$$

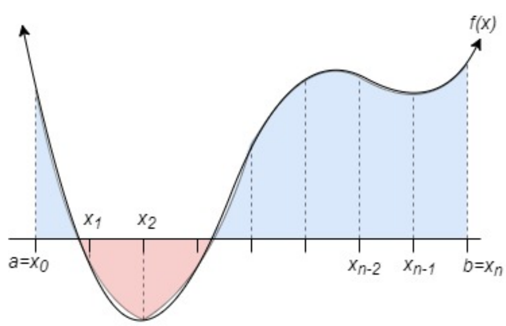
where k is the maximal value of $|f''|$ on the interval.



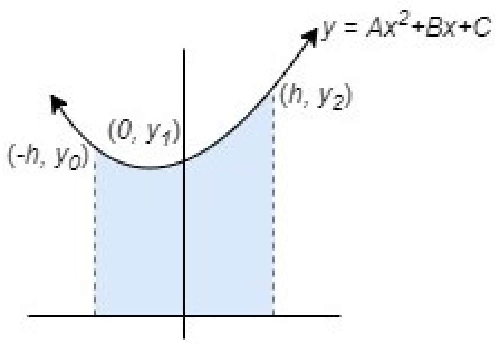
For the function $\sin(x^2)$, $k \approx 60$ on $[0,4]$ and $|E_T| \leq 5$. However, on $[0,2]$ we have that $k \approx 12$ so $|E_T| \leq 0.5$. In either case, we would want to use a much smaller Δx, but you can see the differece the 'smoothness' of the function makes.

Simpson's Rule

To get a better approximation, we can use curves instead of lines by dividing the interval into n subintervals and using $\frac{n}{2}$ parabolas to approximate the function.



To determine the total area we will consider the area of one parabola shifted so that it stradles the y-axis. If we let the parabola have the equation $y = Ax^2 + Bx + C$, then it will have an area of $\frac{h}{3}(2Ah^2 + 6C)$ on the interval $[-h, h]$.



$$\begin{aligned}\int_{-h}^{h} y dx &= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big|_{-h}^{h} \\ &= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch - (-\frac{A}{3}h^3 + \frac{B}{2}h^2 - Ch) \\ &= \frac{2A}{3}h^3 + 2Ch \\ &= \frac{h}{3}(2Ah^2 + 6C)\end{aligned}$$

By using points the parabola passes through, we get the system of equations:

$$\begin{aligned}y_0 &= Ah^2 - Bh + C \\ y_1 &= C \\ y_2 &= Ah^2 + Bh + C\end{aligned}$$

So $y_0 + y_2 = 2Ah^2 + 2C$ implying that

$$2Ah^2 + 6C = y_0 + y_2 + 4C = y_0 + 4y_1 + y_2$$

Thus, the area under the parabola on the interval $[x_k, x_{k+2}]$ is $\frac{\Delta x}{3}(y_k + 4y_{k+1} + y_{k+2})$, so

$$\begin{aligned}\int_a^b f(x)dx \approx \frac{\Delta x}{3} \cdot (f(x_0) + 4f(x_1) &+ f(x_2)) + \\ &\frac{\Delta x}{3} \cdot (f(x_2) + 4f(x_3) + f(x_4)) + \ldots + \\ &\qquad \frac{\Delta x}{3} \cdot (f(x_{n-2}) + 4f(x_{n-1}) + f(x_2n)) \\ = \frac{\Delta x}{3} \cdot (f(x_0) + 4f(x_1) &+ 2f(x_2) + 4(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))\end{aligned}$$

Example 2

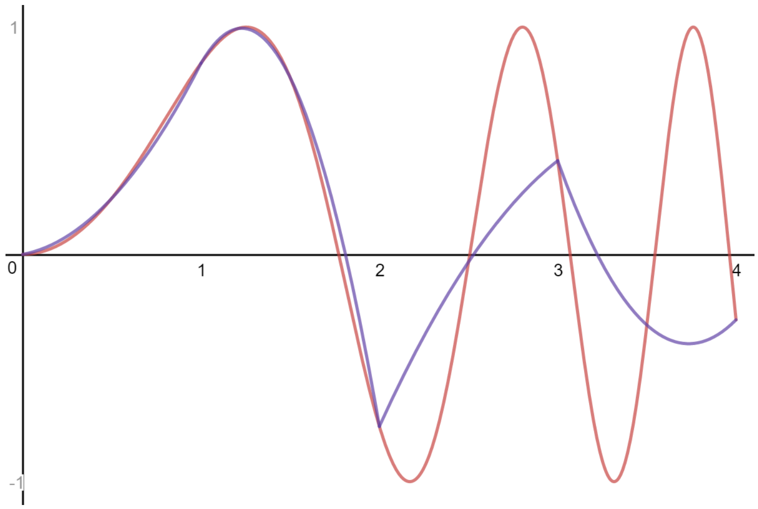
Approximate the area under $f(x) = \sin(x^2)$ on the interval $[0, 4]$ using simspon rule and 8 subintervals (4 parabolas).

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{\Delta x}{3} \cdot (f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)) \\ &= \frac{1}{6} \cdot (\sin(0) + 4\sin(0.25) + 2\sin(1) + \ldots + 4\sin(12.25) + \sin(16)) \\ &= 0.571731...\end{aligned}$$

Again, by examining the graph of $\sin(x^2)$ and Simpson's rule approximation we see that it follows the functions very closesly on $[0, 2]$, but after that the subinterval length is to large for the parabolas to approximate the function. Overall, the approximation is better than trapezoid rule and the error bound reflects that by being proportional to the fourth power of Δx, not the square.

$$|E_S| \leq \frac{b-a}{180}\Delta x^4 k$$

where k is the maximal value of $|f^{(4)}|$ on the interval.



For our function $\sin(x^2)$, $k \approx 3400$ on $[0, 4]$, hence $|E_S| \leq 4.73$; however, $k \approx 165$ on $[0, 2]$ and $|E_S| \leq 0.12$. In both cases, Simpson's rule outperformed Trapezoid rule and that is usually what happens since Simpson's error bound get smaller much faster than the bound for Trapezoid rule.

In case you are curious

$$\int_0^4 \sin(x^2)dx = 0.747134...$$