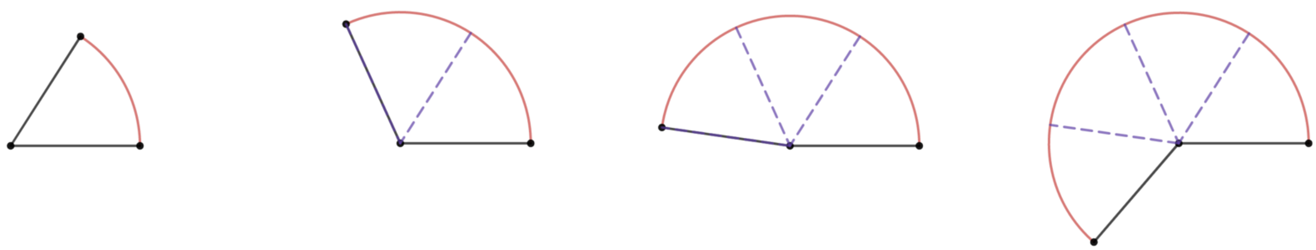
Limits of Trig Functions

Goal: • Understands the importance of the radian when considering trig rates of change.	
Terminology: • Radian	

Discussion: What is a degree? What is a radian?

A degree is defined by cutting a circle into 360 equal pieces. Fairly arbitrary but it does have the property of having many factors so dividing it into halves, thirds, quarters, sixths, etc. is easy.

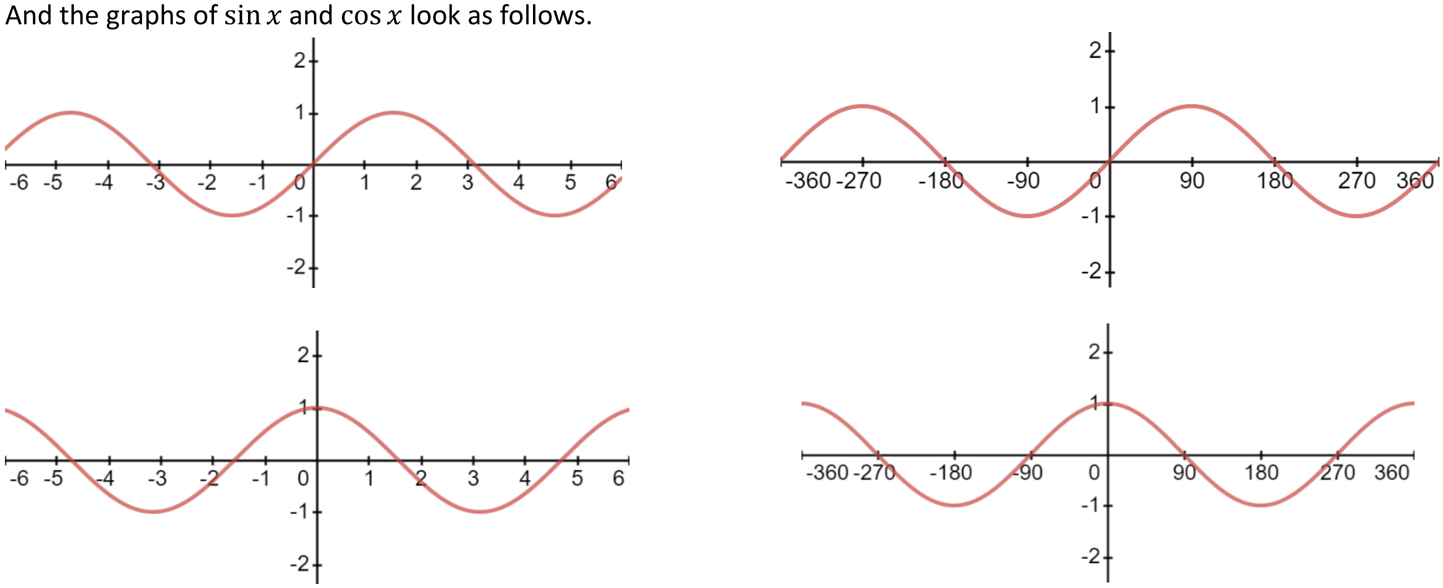
A radian is a measure of the angle that compares the arclength to the radius. This is more natural as it relates the angle to the circle it moves through and is unitless which stops it from being arbitrary.



Recall some important trig identities:

$\sin^2 x + \cos^2 x = 1$	$\tan x = \dfrac{\sin x}{\cos x}$	$\dfrac{1}{\sin x} = \csc x$
$\sin(-x) = -\sin x$	$\dfrac{1}{\tan x} = \cot x$	$\dfrac{1}{\cos x} = \sec x$
$\cos(-x) = \cos x$	$\sin(A+B) = \sin A \cos B + \sin B \cos A$	$\cos(A+B) = \cos A \cos B - \sin A \sin B$

And the graphs of $\sin x$ and $\cos x$ look as follows.



We want to study the calculus of trig functions which begins with limits. No matter the unit of measurement we have that

$$\lim_{x\to 0} \sin x = 0$$

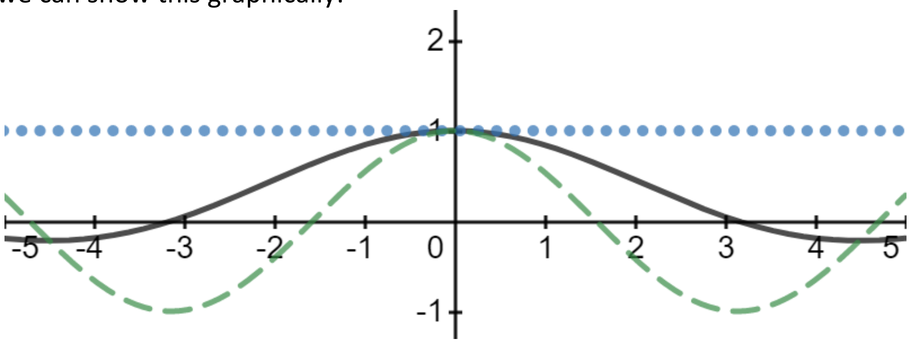
It may seem like it doesn't matter between radian or degree, but for almost every other value of $c \in \mathbb{R}$ we have that

$$\lim_{x\to c\ (\mathrm{rad})} \sin x \neq \lim_{x\to c\ (\mathrm{deg})} \sin x$$

So very clearly the way we measure the angle is important and makes a difference. The limit we are going to investigate is

$$\lim_{x\to 0} \frac{\sin x}{x}$$

Notice that around $x = 0$, when using radians, $\sin x \approx x$, so we should expect the limit to be around 1. Typically this is shown by "squeezing" the function $\frac{\sin x}{x}$ between the functions $y = 1$ and $y = \cos x$ using geometry (whose limits are both 1 as $x \to 0$). But we can show this graphically:



$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Note that this implies $\lim_{x\to 0} \frac{x}{\sin x} = 1$ too.

Example: Evaluate the limit

$$\lim_{x\to 0} \frac{\sin 3x}{x^2 - 3x}$$

$$\lim_{x\to 0} \frac{\sin 3x}{x^2 - 3x} = \lim_{x\to 0} \frac{3 \sin 3x}{3x(x-3)}$$

$$= 3\lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{1}{x-3}$$

$$= 3 \cdot 1 \cdot \frac{1}{-3} = -1$$

Practice: Evaluate the limit

$$\lim_{x\to 0} \frac{\sin Ax}{Bx}$$

$$\lim_{x\to 0} \frac{\sin Ax}{Bx} = \lim_{x\to 0} \frac{A}{B} \cdot \frac{\sin Ax}{Ax}$$

$$= \frac{A}{B} \lim_{x\to 0} \frac{\sin Ax}{Ax}$$

$$= \frac{A}{B} \cdot 1$$

Practice: Evaluate the limit

$$\lim_{x \to 0} \frac{\sin 2x}{\tan 5x}$$

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 2x}{\tan 5x} &= \lim_{x \to 0} \left(\frac{\sin 2x}{\sin 5x} \cdot \cos 5x \right) \\
&= \lim_{x \to 0} \left(\frac{\sin 2x}{x} \cdot \frac{x}{\sin 5x} \cdot \cos 5x \right) \\
&= \lim_{x \to 0} \left(\frac{2 \sin 2x}{2x} \cdot \frac{5x}{5 \sin 5x} \cdot \cos 5x \right) \\
&= \frac{2}{5} \lim_{x \to 0} \left(\frac{\sin 2x}{2x} \cdot \frac{5x}{\sin 5x} \cdot \cos 5x \right) \\
&= \frac{2}{5} \cdot 1 \cdot 1 \cdot 1 = \frac{2}{5}
\end{aligned}$$

Example: Evaluate the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{x}$$

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\
&= \lim_{x \to 0} \frac{\sin^2 x}{x} \cdot \frac{1}{1 + \cos x} \\
&= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \\
&= 1 \cdot \frac{0}{2} = 0
\end{aligned}$$

Practice: Evaluate the limit

$$\lim_{x \to 0} \frac{1 - \sec x}{\sin x}$$

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \sec x}{\sin x} &= \lim_{x \to 0} \frac{1 - \frac{1}{\cos x}}{\sin x} \\
&= \lim_{x \to 0} \frac{\cos x - 1}{\sin x \cos x} \\
&= \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \frac{x}{\sin x} \cdot \frac{-1}{\cos x} \\
&= 0 \cdot 1 \cdot (-1) = 0
\end{aligned}$$

Practice Problems: 7.1 # 1-38 (what you need)

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