Derivative of Sine and Cosine

Goal:
- Can build the derivative of sine and cosine using the definition of the derivative
- Can use derivative rules with basic trig functions

Terminology:
- None

Discussion: Determine the derivative of $\sin x$

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}$$

Because $\sin(x+h)=\sin x\cos h+\sin h\cos x$

$$=\lim_{h\to 0}\frac{\sin x\,(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}$$

$$=0\cdot\sin x+1\cdot\cos x=\cos x$$

Since $\lim_{h\to 0}\frac{\sin h}{h}=1$ and $\lim_{h\to 0}\frac{\cos h-1}{h}=0$

$$\Rightarrow\frac{d}{dx}\sin x=\cos x$$

Note that this means:

$$\int\cos x\,dx=\sin x+C$$

Likewise we can build the derivative of $\cos x$

$$\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{\cos x\cos h-\sin h\sin x-\cos x}{h}$$

Because $\cos(x+h)=\cos x\cos h-\sin h\sin x$

$$=\lim_{h\to 0}\frac{\cos x\,(\cos h-1)}{h}-\lim_{h\to 0}\frac{\sin x\sin h}{h}$$

$$=0\cdot\cos x-1\cdot\sin x=-\sin x$$

Since $\lim_{h\to 0}\frac{\sin h}{h}=1$ and $\lim_{h\to 0}\frac{\cos h-1}{h}=0$

$$\Rightarrow\frac{d}{dx}\cos x=-\sin x$$

And we get that

$$\int\sin x\,dx=-\cos x+C$$

Now we can add trig functions to our derivative rules.

Example: Find $\frac{dy}{dx}$ if:

$$y = e^{\sin x} \cdot \cos^3 x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin x}\right)\cos^3 x + \frac{d}{dx}(\cos^3 x) \cdot e^{\sin x}$$

$$= e^{\sin x} \cdot \cos x \cdot \cos^3 x + 3\cos^2(x) \cdot (-\sin x) \cdot e^{\sin x}$$
$$= e^{\sin x}(\cos^4 x - 3\cos^2 x \sin x)$$

You can stop here but it is worth noting that you can change $\cos^2 x$ to $1 - \sin^2 x$ and if you do that you get a polynomial that you can factor:

$$\Rightarrow \frac{dy}{dx} = e^u(u-1)(u+1)(u^2+3u-1)$$

Where $u = \sin x$

Practice: Find $\frac{dy}{dx}$ if:

$$y = \cos(\sin 3x) - \frac{1}{\sin x}$$

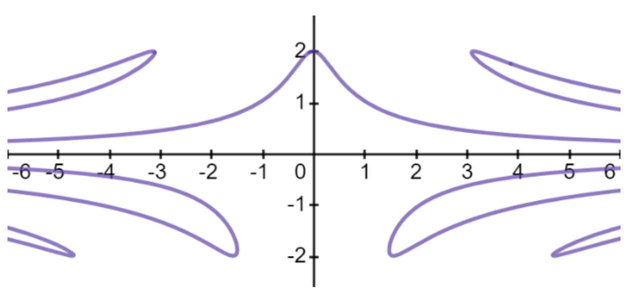
Use chain rule and power rule while changing $\frac{1}{\sin x} = (\sin x)^{-1}$

$$\frac{dy}{dx} = -\sin(\sin 3x) \cdot \cos 3x \cdot 3 + \frac{1}{\sin^2 x} \cdot \cos x$$

$$= -3\cos 3x \cdot \sin(\sin 3x) + \frac{\cos x}{\sin^2 x}$$

Practice: Find $\frac{dy}{dx}$ if

$$2\cos(xy) = y$$



Use implicit differentiation

$$\frac{d}{dx}(2\cos xy) = \frac{d}{dx}y$$
$$-2\sin(xy) \cdot \left(y + x \cdot \frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{dy}{dx}(1 + 2x\sin xy) = -2y\sin xy$$
$$\Rightarrow \frac{dy}{dx} = \frac{-2y\sin xy}{1 + 2x\sin xy}$$

Practice: Find

$$\int \cos(\sin x)\cos x\, dx$$

And evaluate:

$$\int_{\pi/2}^{\pi} \cos(\sin x)\cos x\, dx$$

Use substitution. Let $u = \sin x$, then $du = \cos x\, dx$

So, we get

$$\int \cos(\sin x)\cos x\, dx = \int \cos u\, du$$

Finding an antiderivative of $\cos u$ is easy, we know $\frac{d}{du}\sin u = \cos u$

$$\Rightarrow \int \cos u\, du = \sin u + C$$

Substitute back for x

$$\int \cos(\sin x)\cos x\, dx = \sin(\sin x) + C$$

We can check by taking derivative:

$$\frac{d}{dx}\sin(\sin x) = \cos(\sin x) \cdot \cos x$$

To find the area we just evaluate the antiderivative at the endpoints

$$\int_{\pi/2}^{\pi} \cos(\sin x)\cos x\, dx = \quad \sin(\sin x)\Big|_{\pi/2}^{\pi}$$

$$= \sin(0) - \sin(1)$$
$$= -\sin 1 \approx -0.84$$

Practice Problems: 7.2 # 1-5 (do what you need), 6, 8, 10
11.2 # 2f, 3f
11.3 # 3glnq, 4de