Derivatives of Other Trig Functions

Goal: • Understands that other trig derivatives are built from sine and cosine
Terminology: • $\csc x, \sec x, \cot x$

Discussion: Determine the derivative of $\tan x$

You may think to use limit definition

$$\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

But using the sum of angles for tangent is a little messy: $\tan(x+h)=\frac{\tan x+\tan h}{1-\tan x\tan h}$

Rather, use the identity $\tan x=\frac{\sin x}{\cos x}$ and take derivative of both sides.

$$\Rightarrow \frac{d}{dx}\tan x=\frac{\cos x\cdot\cos x-(-\sin x)\cdot\sin x}{\cos^2 x}$$
$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$
$$=\frac{1}{\cos^2 x}=\sec^2 x$$

$$\frac{d}{dx}\tan x=\sec^2 x$$

And

$$\int \sec^2 x\,dx=\tan x+C$$

Determine the derivatives of $\sec x, \csc x, \cot x$

$\sec x=\frac{1}{\cos x}$	$\csc x=\frac{1}{\sin x}$	$\cot x=\frac{\cos x}{\sin x}$
$\frac{d}{dx}\sec x=-\frac{1}{\cos^2 x}\cdot -\sin x$ $=\frac{\sin x}{\cos^2 x}$ $\frac{d}{dx}\sec x=\sec x\cdot\tan x$ $\int \sec x\tan x\,dx=\sec x+C$	$\frac{d}{dx}\csc x=-\frac{1}{\sin^2 x}\cdot\cos x$ $=-\frac{\cos x}{\sin^2 x}$ $\frac{d}{dx}\csc x=-\csc x\cdot\cot x$ $\int \csc x\cot x\,dx=-\csc x+C$	$\frac{d}{dx}\cot x=\frac{-\sin^2 x-\cos^2 x}{\sin^2 x}$ $=\frac{-1}{\sin^2 x}$ $\frac{d}{dx}\cot x=-\csc^2 x$ $\int \csc^2 x\,dx=-\cot x+C$

Note that all “co” functions have negative derivatives and the similarities between sec/csc and tan/cot

Practice: Find the derivative $\frac{dy}{dx}$

$$y = \sec^2 x + \csc(4x)$$

Use chain rule

$$\frac{dy}{dx} = 2\sec x \cdot \sec x \tan x + \csc(4x)\cot(4x) \cdot 4$$
$$= 2\sec^2 x \tan x + 4\csc 4x \cdot \cot 4x$$

Practice: Linearize the function

$$f(x) = \tan\left(\frac{1}{2}x\right) + 1$$

About the point $x = 0$

Find the tangent line at $x = 0$

$$f'(x) = \frac{1}{2}\sec^2\left(\frac{1}{2}x\right)$$

$$f'(0) = \frac{1}{2}, \qquad f(0) = 1$$

$$\Rightarrow L(x) = \frac{1}{2}x + 1$$

Practice: Find the two solutions to

$$x^2 = \cot x, \qquad x \in (-\pi, \pi)$$

Use Newton's Method to find the zeros of

$$f(x) = x^2 - \cot x$$
$$\Rightarrow f'(x) = 2x + \csc^2 x$$

Then solve for x using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x = A - \frac{A^2 - \cot A}{2A + \csc^2 A}$$

If $A = 3$ then we find $x = 0.89520604 \ldots$

If $A = -3$ then we find $x = -3.03333516 \ldots$

Practice: Solve the differential equation

$$\frac{dy}{dx} = \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}, \qquad y(0) = 1$$

We want to find a function $y(x)$ such that $y'(x) = \sec x$

$$y(x) = \int \sec x \, dx$$

Use the unitary fraction, let $u = \sec x + \tan x$ so $du = (\sec x \tan x + \sec^2 x)dx$

$$y(x) = \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \right) dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

Substitute back into x

$$y(x) = \ln|\sec x + \tan x| + C$$

Solve for $y(0) = 1$ so we get $1 = \ln|\sec 0 + \tan 0| + C \Rightarrow 1 = C$

$$y(x) = \ln|\sec x + \tan x| + 1$$

Practice Problems: 7.3 # 1-3 (do what you need), 4, 8, 12, 13
11.2 # 1op, 2gh
11.3 # 3e, 5