Derivatives of Inverse Trig Functions

Goal:
• Understands how to find the derivative of $\arcsin x$ and $\arctan x$
Terminology:
• "Arc"-function

Discussion: What is $\sin(\arcsin x)$? What is $\cos(\arcsin x)$?

If $y = \sin x$ takes an angle x and outputs the ratio of side lengths. Then $y = \arcsin x$ does the inverse and takes a ratio x and outputs an angle y. Not that in radians the angle y is equal to the arclength on the unit circle, hence arc sine.

$$\sin(\arcsin x) = x$$

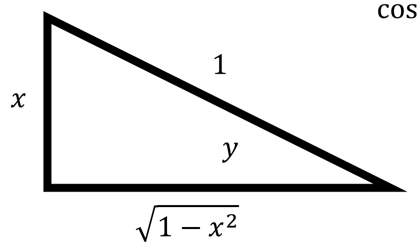
Since arcsin and sine are inverses.

If we consider $y = \arcsin x$ then the ratio of opposite to hypotenuse sides is $\frac{x}{1}$ and the angle is y. Then the adjacent side is $\sqrt{1-x^2}$ and so

$$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Alternatively, note that $\cos x = \sqrt{1-\sin^2 x}$ so

$$\cos(\arcsin x) = \sqrt{1-\sin^2(\arcsin x)} = \sqrt{1-x^2}$$



We are going to determine the derivative of arcsin and arctan (arguably the most important inverse trig functions)

Example: Find

$$\frac{d}{dx}\arcsin x$$

Set $y = \arcsin x$ and then take the sine of both sides $\sin y = x$. Find $\frac{d}{dx}$ implicitly

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Substitute for y:

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Practice: Find

$$\frac{d}{dx}\arctan x$$

Set $y = \arctan x$ and then take the tangent of both sides and find $\frac{d}{dx}$ implicitly

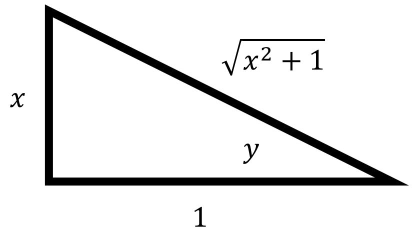
$$y = \arctan x \Rightarrow \tan y = x$$

$$\sec^2 y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \cos^2 y$$

$$= \cos^2 \arctan x$$

Using a triangle, if $\tan y = x$ then the opposite side is x and the adjacent side is 1, so the hypotenuse is $\sqrt{x^2+1}$



$$\cos y = \frac{1}{\sqrt{x^2+1}}$$

$$\Rightarrow \frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

$$\Rightarrow \int \frac{dx}{x^2+1} = \arctan x + C$$

Practice: Determine $\frac{dy}{dx}$ for the following function

$$y = \arcsin\left(\sqrt{x}\right) + \arctan(2x)$$

Use chain rule

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} + \frac{2}{4x^2+1}$$

$$= \frac{1}{2\sqrt{x-x^2}} + \frac{2}{4x^2+1}$$

Example: Solve the antiderivative

$$\int \frac{3}{2+4x^2}dx$$

Recognize that it has the form $\frac{a}{1+(bx)^2}$ which is close to arctan. Factor out the 3 and the 2

$$\int \frac{3}{2+4x^2}dx = \frac{3}{2}\int \frac{dx}{1+2x^2}$$

We want $2x^2$ to be a perfect square so let $u^2 = 2x^2 \Rightarrow 2udu = 4xdx$

$$= \frac{3}{2}\int \frac{1}{1+u^2} \cdot \frac{2udu}{4x}$$

We have $u = \sqrt{2}x$

$$= \frac{3}{2}\int \frac{1}{1+u^2} \cdot \frac{1}{\sqrt{2}}du = \frac{3}{2\sqrt{2}}\arctan u + C$$

$$= \frac{3}{2\sqrt{2}}\arctan \sqrt{2}x + C$$

Practice: Solve the antiderivative

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$$

This form looks a bit like arcsin where we have $\frac{a}{\sqrt{1-(bx)^2}}$. We want the e^{2x} to be a perfect square so let $u^2 = e^{2x}$, then $udu = e^{2x}dx$. Note that $u = e^x$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{u}{\sqrt{1-u^2}} \cdot \frac{u}{e^{2x}}du$$

$$= \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{u^2}{u^2}du$$

$$= \arcsin u + C$$

$$= \arcsin e^x + C$$

Practice: Solve the antiderivative

$$\int \frac{\sqrt{x}}{1+x^3}dx$$

Recognize that this kinda looks like it has the form $\frac{a}{1+(bx)^2}$ which is close to arctan. It's not there yet but we want x^3 to be a perfect square so let $u^2 = x^3$ that way $2udu = 3x^2dx$

$$\int \frac{\sqrt{x}}{1+x^3}dx = \int \frac{\sqrt{x}}{1+u^2} \cdot \frac{2u}{3x^2}du$$

We have $\frac{\sqrt{x}}{x^2} = x^{-\frac{3}{2}} = \frac{1}{u}$

$$= \int \frac{1}{1+u^2} \cdot \frac{2u}{3u}du$$

$$= \frac{2}{3}\arctan u + C$$

$$= \frac{2}{3}\arctan x^{\frac{3}{2}} + C$$

Practice Problems: 7.6 # 1 (do what you need but skip $\cos^{-1} x$), 2-5
11.2 # 3cde
11.3 # 3r