Trig Identities and Their Use in Calculus

Pythagorean Identities

- $\cos^2 x + \sin^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \csc^2 x$

Derived from the coordinates on a unit circle:

 $x^2 + y^2 = 1 \Longrightarrow \cos^2 \theta + \sin^2 \theta = 1$

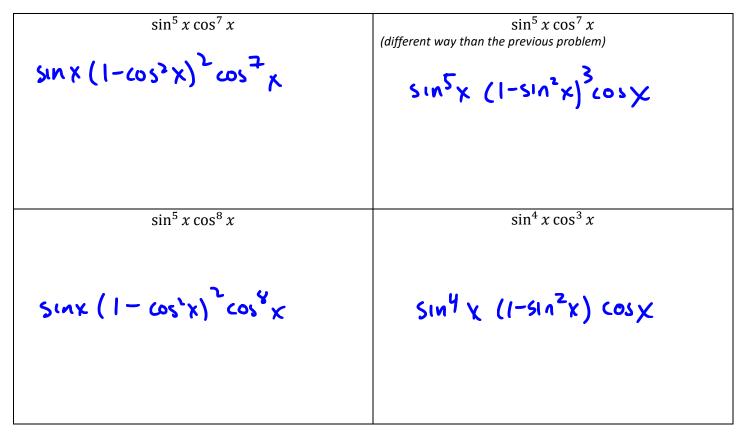
and then divide both sides by $\sin^2 \theta$ or $\cos^2 \theta$ to get the other two.

Usage in Calculus:

Aside from the general times you will often see $\sin^2 x + \cos^2 x$ and replace with 1, these are vital to solve integrals.

We can replace $\sin^3 x \cos^n x = \sin x \cdot (1 - \cos^2 x) \cdot \cos^n x$ which has 1 sine function which is really nice or $\cos^5 x \sin^m x = \cos x \cdot (1 - \sin^2 x) \sin^m x$ which also only has 1 cosine function which will be really nice.

Practice: Use the Pythagorean trig identity to put the following in terms of a single sine or cosine function times a bunch of the other.



Also, in calculus, $\tan^2 x$ is not as nice as $\sec^2 x$, so oftentimes we will replace $\tan^2 x$ with $\sec^2 x - 1$.

There is also the very nice property of these identities that they take a sum of squares and turn it into another square. Multiplying by a constant square we get

- $a^2 \sin^2 x + a^2 \cos^2 x = a^2$
- $a^2 \tan^2 x + a^2 = a^2 \sec^2 x$

From these we get three general shapes:

$$(a \cdot \cos x)^2 = a^2 - (a \cdot \sin x)^2 \Longrightarrow \frac{v^2}{2} = a^2 - u^2$$
$$(a \cdot \sec x)^2 = a^2 + (a \cdot \tan x)^2 \Longrightarrow \frac{v^2}{2} = a^2 + u^2$$
$$(a \cdot \tan x)^2 = (a \cdot \sin x)^2 - a^2 \Longrightarrow \frac{v^2}{2} = u^2 - a^2$$

Hence, we can make a trig substitution to change a sum or difference of squares into a perfect square. For example: $\sqrt{4 - x^2}$ we can set $x = 2 \sin \theta$ then we get $\sqrt{4 - 4 \sin^2 \theta} = 2 |\cos \theta|$

Practice: Use Pythagorean trig identities to reduce the following into a perfect square. You may need to complete the square to do so.

$\sqrt{9-x^2}$ let x = 3 cos θ (or 3 sm θ) $\sqrt{9-9} \cos^2 \theta = \sqrt{9} \sin^2 \theta$ $= 31 \sin \theta$	$\sqrt{x^2 - 16}$ let $x = 4 \sec \theta$ $\sqrt{16 \sec^2 \theta} - 16 = \sqrt{16 \tan^2 \theta}$ $= 41 \tan \theta$
$\sqrt{x^2 + 25}$ Let x = 5 tan 0 $\sqrt{25 \tan^2 \theta + 25} = \sqrt{25 \sec^2 \theta}$ $= 5 \sec \theta $	$\sqrt{x^2 - 2x + 5}$ $\sqrt{(x-1)^2 + 4} \text{let } x-1 = 2 \tan \theta$ $\sqrt{4 \tan^2 \theta + 4} = \sqrt{4 \sec^2 \theta}$ $= 2 \sec \theta $

$$\frac{-(x^{2}-8x)-7}{\sqrt{-x^{2}+8x-7}}$$

$$\frac{\sqrt{-x^{2}+8x-7}}{\sqrt{-x^{2}+8x-7}}$$

$$\frac{\sqrt{-16}}{\sqrt{-16}} = \frac{16}{2} + \frac{16}{$$

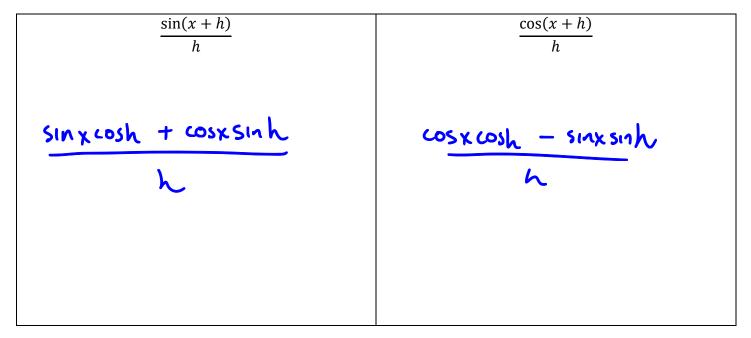
Sum of Angles

- sin(A + B) = sin A cos B + sin B cos A
- $\cos(A+B) = \cos A \cos B \sin A \sin B$

Derived from looking at a compound of two triangles (see videos)

Usage in Calculus:

This will be used early in calculus to derive the derivative of sine and cosine. Use the identity above to separate the following expressions into a sum of sine and cosines.



Double Angles

- $\sin(2A) = 2\sin A \cos A$
- $\cos(2A) = \cos^2 A \sin^2 A$

Derived from the sum of angles by letting B = A.

Note that $\cos 2A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$ by using the Pythagorean trig identity.

Usage in Calculus:

We typically use the half angle formulas (rearranged versions of the cosine identity) that allow us to reduce the power of sine and cosine.

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$
$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

For example: it is beneficial to change $\sin^2 x \cos^2 x = \frac{1}{4}\sin^2 2x = \frac{1}{8}(1 - \cos 4x)$ or we can change to $\cos^4 x = \frac{1}{4}(1 + \cos 2x)^2 = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x) = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}(1 + \cos 4x)$

Practice: Use the double angle identity of sine and cosine to reduce the powers to a single power of sine or cosine OR until we get an odd power of sine or cosine and then use a Pythagorean identity.

$$\begin{aligned} \sin^{4} x \\ \left(\sin^{2} \kappa \right)^{2} &= \left(\frac{1 - (\cos^{2} \lambda)^{2}}{2} \right)^{2} \\ &= \frac{1}{9} \left(1 - 2\cos^{2} \chi + \cos^{2} 2 \chi \right) \\ &= \frac{1}{9} \left(1 - 2\cos^{2} \chi + \cos^{2} 2 \chi \right) \\ &= \frac{1}{9} \left(1 - 2\cos^{2} \chi + \frac{1 + \cos^{2} \chi}{2} \right) \\ &= \frac{1}{8} \left(1 + \cos^{2} \chi - \cos^{2} 2 \chi - \cos^{2} 2 \chi \right) \\ &= \frac{1}{8} \left(1 + \cos^{2} \chi - \cos^{2} 2 \chi - \cos^{2} 2 \chi \right) \\ &= \frac{1}{8} \left(1 + \cos^{2} \chi - (\frac{1 + \cos^{4} \chi}{2}) - \cos^{2} \chi \right) \\ &= \frac{1}{8} \left(1 + \cos^{2} \chi - (\frac{1 + \cos^{4} \chi}{2}) - \cos^{2} \chi \right) \\ &= \frac{1}{8} - \frac{1}{16} - \frac{\cos^{4} \chi}{16} + \frac{1}{8} \cos^{2} \chi \sin^{2} \chi \end{aligned}$$

$$\frac{\sin^{4} x \cos^{2} x}{\left(1 - \cos^{2} x\right)^{2} \left(\frac{1 + \cos^{2} x}{2}\right)} = \frac{1}{8} \left(1 - \cos^{2} x - \cos^{2} 2x + \cos^{2} 2x\right)} = \frac{1}{16} \left[\frac{3}{8} - \frac{1}{2} \cos^{4} x + \frac{1}{8} \cos^{8} x\right]$$

$$= \frac{1}{16} \left[\frac{3}{8} - \frac{1}{2} \cos^{4} x + \frac{1}{8} \cos^{8} x\right]$$

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$$\int \int \frac{1}{8} \left(1 - \cos^{4} x - \frac{1}{8} \cos^{2} x \sin^{2} x\right)$$

$$= \frac{1}{8} \left(1 - \cos^{4} x - \frac{1}{8} \cos^{2} x \sin^{2} x\right)$$

$$= \frac{1}{8} \left(1 - 3\cos^{2} x + 3\cos^{2} 2x - \cos^{3} 2x\right)$$

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