Trig Identities and Their Use in Calculus

Pythagorean Identities

- $\cos^2 x + \sin^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \csc^2 x$

Derived from the coordinates on a unit circle:

$$x^2 + y^2 = 1 \Longrightarrow \cos^2\theta + \sin^2\theta = 1$$

and then divide both sides by $\sin^2\theta$ or $\cos^2\theta$ to get the other two.

Usage in Calculus:

Aside from the general times you will often see $\sin^2 x + \cos^2 x$ and replace with 1, these are vital to solve integrals.

We can replace $\sin^3 x \cos^n x = \sin x \cdot (1 - \cos^2 x) \cdot \cos^n x$ which has 1 sine function which is really nice or $\cos^5 x \sin^m x = \cos x \cdot (1 - \sin^2 x) \sin^m x$ which also only has 1 cosine function which will be really nice.

Practice: Use the Pythagorean trig identity to put the following in terms of a single sine or cosine function times a bunch of the other.

$\sin^5 x \cos^7 x$ $\sin x\,(1-\cos^2 x)^2\cos^7 x$	$\sin^5 x \cos^7 x$ *(different way than the previous problem)* $\sin^5 x\,(1-\sin^2 x)^3\cos x$
$\sin^5 x \cos^8 x$ $\sin x\,(1-\cos^2 x)^2\cos^8 x$	$\sin^4 x \cos^3 x$ $\sin^4 x\,(1-\sin^2 x)\cos x$

Also, in calculus, $\tan^2 x$ is not as nice as $\sec^2 x$, so oftentimes we will replace $\tan^2 x$ with $\sec^2 x - 1$.

There is also the very nice property of these identities that they take a sum of squares and turn it into another square. Multiplying by a constant square we get

- $a^2 \sin^2 x + a^2 \cos^2 x = a^2$
- $a^2 \tan^2 x + a^2 = a^2 \sec^2 x$

From these we get three general shapes:

$$(a \cdot \cos x)^2 = a^2 - (a \cdot \sin x)^2 \Longrightarrow v^2 = a^2 - u^2$$

$$(a \cdot \sec x)^2 = a^2 + (a \cdot \tan x)^2 \Longrightarrow v^2 = a^2 + u^2$$

$$(a \cdot \tan x)^2 = (a \cdot \sin x)^2 - a^2 \Longrightarrow v^2 = u^2 - a^2$$

Hence, we can make a trig substitution to change a sum or difference of squares into a perfect square. For example: $\sqrt{4 - x^2}$ we can set $x = 2 \sin \theta$ then we get $\sqrt{4 - 4\sin^2 \theta} = 2|\cos \theta|$

Practice: Use Pythagorean trig identities to reduce the following into a perfect square. You may need to complete the square to do so.

$\sqrt{9 - x^2}$	$\sqrt{x^2 - 16}$
let $x = 3\cos\theta$ (or $3\sin\theta$) $\sqrt{9 - 9\cos^2\theta} = \sqrt{9\sin^2\theta}$ $= 3\|\sin\theta\|$	let $x = 4\sec\theta$ $\sqrt{16\sec^2\theta - 16} = \sqrt{16\tan^2\theta}$ $= 4\|\tan\theta\|$
$\sqrt{x^2 + 25}$	$\sqrt{x^2 - 2x + 5}$
let $x = 5\tan\theta$ $\sqrt{25\tan^2\theta + 25} = \sqrt{25\sec^2\theta}$ $= 5\|\sec\theta\|$	$\sqrt{(x-1)^2 + 4}$ let $x - 1 = 2\tan\theta$ $\sqrt{4\tan^2\theta + 4} = \sqrt{4\sec^2\theta}$ $= 2\|\sec\theta\|$

$-(x^2-8x)-7$

$\sqrt{x^2+4x-12}$	$\sqrt{-x^2+8x-7}$
$\sqrt{(x+2)^2-16}$ let $x+2=4\sec\theta$ $\sqrt{16\sec^2\theta-16}=4\lvert\tan\theta\rvert$	let $x-4=3\sin\theta$ $\sqrt{-(x-4)^2+9}$ $\sqrt{9-9\sin^2\theta}=3\lvert\cos\theta\rvert$

Sum of Angles

- $\sin(A+B)=\sin A\cos B+\sin B\cos A$
- $\cos(A+B)=\cos A\cos B-\sin A\sin B$

Derived from looking at a compound of two triangles (see videos)

Usage in Calculus:

This will be used early in calculus to derive the derivative of sine and cosine. Use the identity above to separate the following expressions into a sum of sine and cosines.

$\frac{\sin(x+h)}{h}$	$\frac{\cos(x+h)}{h}$
$\frac{\sin x\cos h+\cos x\sin h}{h}$	$\frac{\cos x\cos h-\sin x\sin h}{h}$

Double Angles

- $\sin(2A) = 2\sin A \cos A$
- $\cos(2A) = \cos^2 A - \sin^2 A$

Derived from the sum of angles by letting $B = A$.

Note that $\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1$ by using the Pythagorean trig identity.

Usage in Calculus:

We typically use the half angle formulas (rearranged versions of the cosine identity) that allow us to reduce the power of sine and cosine.

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

For example: it is beneficial to change $\sin^2 x \cos^2 x = \frac{1}{4}\sin^2 2x = \frac{1}{8}(1 - \cos 4x)$ or we can change to $\cos^4 x = \frac{1}{4}(1 + \cos 2x)^2 = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x) = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}(1 + \cos 4x)$

Practice: Use the double angle identity of sine and cosine to reduce the powers to a single power of sine or cosine OR until we get an odd power of sine or cosine and then use a Pythagorean identity.

$\sin^4 x$	$\sin^2 x \cos^4 x$
$(\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2$ $= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$ $= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)$ $= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$	$(1-x)(1+x)^2 = (1-x^2)(1+x) = 1+x-x^2-x^3$ $\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2$ $= \frac{1}{8}(1 + \cos 2x - \cos^2 2x - \cos^3 2x)$ $= \frac{1}{8}\left(1 + \cancel{\cos 2x} - \left(\frac{1+\cos 4x}{2}\right) - \cos 2x\,(\cancel{1} - \sin^2 2x)\right)$ $= \frac{1}{8} - \frac{1}{16} - \frac{\cos 4x}{16} + \frac{1}{8}\cos 2x \sin^2 2x$

$\sin^4 x \cos^2 x$

$$\left(\frac{1-\cos 2x}{2}\right)^2\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{8}\left(1-\cos 2x-\cos^2 2x+\cos^3 2x\right)$$

$$= \frac{1}{8}\left(1-\cancel{\cos 2x}-\left(\frac{1+\cos 4x}{2}\right)+\cos 2x \, (\cancel{1}-\sin^2 2x)\right)$$

$$= \frac{1}{8}-\frac{1}{16}-\frac{1}{16}\cos 4x-\frac{1}{8}\cos 2x\sin^2 2x$$

$\sin^4 x \cos^4 x$

$$(\sin x\cos x)^4 = \left(\frac{1}{2}\sin 2x\right)^4$$

$$= \frac{1}{16}\sin^4 2x$$

$$= \frac{1}{16}\left[\frac{3}{8}-\frac{1}{2}\cos 4x+\frac{1}{8}\cos 8x\right]$$

$\downarrow$

Since we know $\sin^4 x$

$\sin^6 x$

$$\sin^6 x = \left(\frac{1-\cos 2x}{2}\right)^3 = \frac{1}{8}\left(1-3\cos 2x+3\cos^2 2x-\cos^3 2x\right)$$

$$= \frac{1}{8}\left(1-3\cos 2x+3\left(\frac{1+\cos 4x}{2}\right)-\cos 2x\,(1-\sin^2 2x)\right)$$

$$= \frac{1}{8}+\frac{3}{2}-\frac{1}{2}\cos 2x+\frac{3}{16}\cos 4x+\frac{1}{8}\cos 2x\sin^2 2x$$