6.2 -Antiderivatives using u-substitution

Mr. Guillen's AP Calculus

Recall that we definied the antiderivative of f as

$$\int f(x)dx$$

To find this antiderivative, we had to work backwords to find a function F such that $\frac{d}{dx}F = f$. This is easy enough when the function is not that complex and has a basic antiderivative. By basic, I mean a single polynomial, exponenential, or trig function. For reference, these are the basic antiderivatives you need to be familiar with.

$$\frac{d}{dx}x^n = nx^{n-1} \implies \int x^{n-1}dx = \frac{x^n}{n}, n \neq 0 \qquad \frac{d}{dx}\ln x = \frac{1}{x} \implies \int \frac{1}{x}dx = \ln x$$

$$\frac{d}{dx}b^x = b^x \ln b \implies \int b^x dx = \frac{b^x}{\ln b}$$

$$\frac{d}{dx}\sin x = \cos x \implies \int \cos x dx = \sin x \qquad \frac{d}{dx}\csc x = -\csc x \cot x \implies \int \csc x \cot x dx = -\csc x$$

$$\frac{d}{dx}\cos x = -\sin x \implies \int \sin x dx = -\cos x \qquad \frac{d}{dx}\sec x = \sec x \tan x \implies \int \sec x \tan x dx = \sec x$$

$$\frac{d}{dx}\tan x = \sec^2 x \implies \int \sec^2 x dx = \tan x \qquad \frac{d}{dx}\cot x = -\csc^2 x \implies \int \csc^2 x dx = -\cot x$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \implies \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x \qquad \frac{d}{dx}\arctan x = \frac{1}{x^2+1} \implies \int \frac{1}{x^2+1}dx = \arctan x$$

Almost every integral we consider for the remainder of the course will rely on these basic antiderivatives in some form. The focus in this lesson is to consider integrals of the form:

$$\int f(g(x))g'(x)dx$$

Our motive for looking at this specific type of integrals is that if we let $u = g(x)$ than $du/dx = g'(x)$ which implies $du = g'(x)dx$. Using this substitution, we can reduce the integral of a composition of functions to simply

$$\int f(u)du$$

and find a simple antiderivative. The big idea is that if we see the derivative of some part of the function in the integral, or a composition of functions, we should try using u-substituition.

Example 1 - Straight substitution

$$\int 3\sec^2(3x-4)dx$$

We notice that the function $3x-4$ is composed inside the function $\sec^2 x$, and its derivative, 3, appears in the integrand. Therefore, we can let $u = 3x-4$ and $du = 3dx$ which means

$$\int 3\sec^2(3x-4)dx = \int \sec^2 u du = \tan u + C = \tan(3x-4) + C$$

However, since constants can multiply through an integral, we don't need to see the exact derivative present, just some mutiple of it.

Example 2 - Straight substitution

$$\int \frac{x^2}{x^3+1}dx$$

Notice that the derivative of x^3+1 is $3x^2$ and x^2dx appears in the integral. We could either multiply the integral by $\frac{3}{3}$ to get $3x^2$ or we could use algebra to see that if $u = x^3+1$ then $du = 3x^2dx$ which implies $x^2dx = \frac{1}{3}du$. Therefore

$$\int \frac{x^2}{x^3+1}dx = \int \frac{\frac{1}{3}du}{u} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln u + C = \frac{1}{3}\ln(x^3+1) + C$$

Be aware that when we make our choice of u, we need to make everything in terms of u.

Example 3 - Multiple substitutions

$$\int x^3\sqrt{x^2+1}dx$$

There are 3 possible choices we might consider for u. We may think that $u = x^3$ would be good since x^2 appears, but this will not work well since it is inside another function. We could let $u = \sqrt{x^2+1}$, in that case $u^2 = x^2+1$ and so $2udu = 2xdx$ or equivalently $udu = xdx$. Here we can evaluate the integral since we can determine x^2 in terms of u.

$$\begin{aligned}
\int x^3\sqrt{x^2+1}dx &= \int x^2\sqrt{x^2+1}xdx \\
&= \int (u^2-1)(u)udu \\
&= \int (u^4-u^2)du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C
\end{aligned}$$

Otherwise, we might have let $u = x^2+1$ as it looks to be inside another function. In this case $du = 2xdx$ and again we are good to continue since we will have an x^2 left over that we can put in terms of u.

$$\begin{aligned}
\int x^3\sqrt{x^2+1}dx &= \int x^2\sqrt{x^2+1}xdx \\
&= \int (u-1)\sqrt{u}\frac{du}{2} \\
&= \frac{1}{2}\int (u^{3/2}-u^{1/2})du \\
&= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\
&= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C
\end{aligned}$$

As is true in many areas of mathematics, there is no 'one way' to use u-substitution correctly. If you see something, try it and see what happens. A lot of the time it will work out and even if it doesn't you will learn from why it didn't.

Example 4 - Building a familiar function

$$\int \frac{1}{\sqrt{4-x^2}}dx$$

On the surface, there doesn't seem to be a good choice of u. If we let $u = 4 - x^2$ or $\sqrt{4-x^2}$ then we don't have xdx needed for du, and if we let $u = x$ nothing is changed. However, we can see this looks a bit like $\dfrac{1}{\sqrt{1-x^2}}$ which is the derivative of $\arcsin(x)$. Thus, we are motivated to let $x = 2u$ and $dx = 2du$.

$$\begin{aligned}\int \frac{1}{\sqrt{4-x^2}}dx &= \int \frac{2}{\sqrt{4-4u^2}}du \\ &= \int \frac{2}{2\sqrt{1-u^2}}du \\ &= \arcsin(u) + C \\ &= \arcsin\left(\frac{x}{2}\right) + C\end{aligned}$$

Example 5 - Simplifying or changing the integrand

$$\int \sin^2 x dx$$

Again, there is no good choice of u so we need to change the integrand to something else. Using a trig identity, we know $\sin^2 x = \frac{1-\cos 2x}{2}$ so,

$$\begin{aligned}\int \sin^2 x dx &= \int \frac{1-\cos 2x}{2}dx \\ &= \int \frac{1}{2}dx - \int \frac{\cos 2x}{2}dx \\ &= \frac{x}{2} - \frac{\sin 2x}{4} + C\end{aligned}$$

Example 6 - Discrete integral

$$\int_1^e \frac{\ln(x^2)}{x}dx$$

We proceed as normal but remember we are integrating along x (hence the dx), so when we make our substitution the interval will change too. To begin, we see that the derivative of $\ln x$ is present, but in order to let $u = \ln x$, we must first simplify $\ln x^2 = 2\ln x$. Then we get that $du = dx/x$ and rather than integrating on the interval $x \in [1, e]$, we are instead integrating along the interval $u \in [0, 1]$ since $\ln 1 = 0$ and $\ln e = 1$.

$$\begin{aligned}\int_1^e \frac{\ln(x^2)}{x}dx &= \int_0^1 2udu \\ &= u^2\Big|_0^1 \quad \textbf{OR} \quad \ln^2 x\Big|_1^e \\ &= 1\end{aligned}$$

Notice if we substituted back in terms of x, we would not have to change the interval. Both answers are the same since we are consitent when we integrate with respect to u and when we integrate with respect to x.