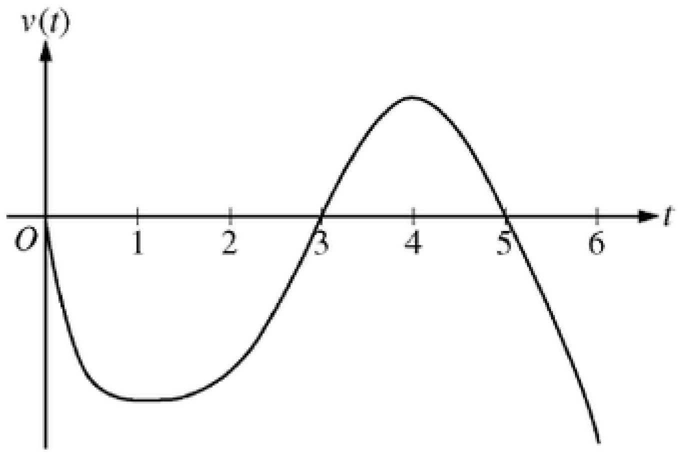
# Week 5 Motion Review



Graph of $v$

A particle moves along the $x$-axis so that its velocity at time $t$, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the $t$-axis and the graph of $v$ on the intervals [0,3], [3,5], and [5,6] are 8, 3, and 2 respectively. At time $t = 0$, the particle is $x = -2$.

**1.** During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

Please respond on separate paper, following directions from your teacher.

**Part D**

2 points can be earned.

1 point for answer.

1 point for justification.

The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.

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| 0 | 1 | 2 |
|---|---|---|

The student response earns two of the following points:

2 points can be earned.

1 point for answer.

1 point for justification.

The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.

**2.** On the interval 2 < t < 3 , is the speed of the particle increasing or decreasing? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point can be earned.

1 point for answer with reason.

The speed is decreasing on the interval $2 < t < 3$ since on the interval $v < 0$ and v is increasing .

| 0 | 1 |
|---|---|

The student response earns one of the following points:

1 point can be earned.

1 point for answer with reason.

The speed is decreasing on the interval $2 < t < 3$ since on the interval $v < 0$ and v is increasing .

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Particle *X* moves along the positive *x*-axis so that its position at time $t \geq 0$ is given by $x(t) = 5t^3 - 9t^2 + 7$

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**3.** Is particle *X* moving toward the left or toward the right at time $t = 1$ ? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

**Part A**

The response can earn up to 2 points:

1 point: For consideration of $x'(1)$

1 point: Correct answer with reason.

$x'(t) = 15t^2 - 18t$

$x'(1) = 15 - 18 = -3$

Since $x'(1) < 0$, the particle is moving to the left at time $t = 1$

| 0 | 1 | 2 |
|---|---|---|

The response earns both of the following points:

1 point: For consideration of $x'(1)$

1 point: Correct answer with reason.

$x'(t) = 15t^2 - 18t$

$x'(1) = 15 - 18 = -3$

Since $x'(1) < 0$, the particle is moving to the left at time $t = 1$.

**4.** A second particle, *Y*, moves along the positive *y*-axis so that its position at time *t* is given by $y(t) = 7t + 3$ . At any time $t, t \geq 0$ , the origin and the positions of the particles *X* and *Y* are the vertices of a triangle in the first quadrant. Find the rate of change of the area of the triangle at time $t=1$ . Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

**Part C**

The response can earn up to 4 points:

1 point: For the area function

$$A(t) = \frac{1}{2}x(t)y(t)$$
$$= \frac{1}{2}(5t^3 - 9t^2 + 7)(7t + 3)$$

2 point: For the derivative

$$A'(t) = \frac{1}{2}\left[(15t^2 - 18t)(7t + 3) + (5t^3 - 9t^2 + 6)(7)\right]$$

1 point: For the correct answer

$$A'(1) = \frac{1}{2}[(-3)(10) + (3)(7)] = \frac{1}{2}[-30 + 21] = -\frac{9}{2}$$

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|

The response earns all four of the following points:

1 point: For the area function

$$A(t) = \frac{1}{2}x(t)y(t)$$
$$= \frac{1}{2}(5t^3 - 9t^2 + 7)(7t + 3)$$

2 point: For the derivative

$$A'(t) = \frac{1}{2}\left[(15t^2 - 18t)(7t + 3) + (5t^3 - 9t^2 + 6)(7)\right]$$

1 point: For the correct answer

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$$A'(1) = \frac{1}{2}[(-3)(10) + (3)(7)] = \frac{1}{2}[-30 + 21] = -\frac{9}{2}$$

**5.** At what time $t \geq 0$ is particle $X$ farthest to the left? Justify your answer.

Please respond on separate paper, following directions from your teacher.

**Part B**

The response can earn up to 3 points:

1 point: Consideration of $x'(t)=0$

1 point: Identification of $t=\frac{6}{5}$

1 point: Correct answer with reason

$x'(t)=3t(5t-6)=0 \Rightarrow t=0,\ t=\frac{6}{5}$

Since $x'(t)<0$, for $0<t<\frac{6}{5}$ and $x'(t)>0$ for $t>\frac{6}{5}$, the particle is farthest to the left at time $t=\frac{6}{5}$

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The response earns all three of the following points:

1 point: Consideration of $x'(t)=0$

1 point: Identification of $t=\frac{6}{5}$

1 point: Correct answer with reason

$x'(t)=3t(5t-6)=0 \Rightarrow t=0,\ t=\frac{6}{5}$

Since $x'(t)<0$, for $0<t<\frac{6}{5}$ and $x'(t)>0$ for $t>\frac{6}{5}$, the particle is farthest to the left at time $t=\frac{6}{5}$

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| $t$ (minutes) | 0 | 12 | 20 | 24 | 40 |
|---|---|---|---|---|---|
| $v(t)$ (meters per minute) | 0 | 200 | 240 | −220 | 150 |

Johanna jogs along a straight path. For $0 \le t \le 40$ , Johanna's velocity is given by a differentiable function $v$. Selected values of $v(t)$, where $t$ is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

**6.** Bob is riding his bicycle along the same path. For $0 \le t \le 10$ , Bob's velocity is modeled by $B(t)= t^3 - 6t^2 + 300$, where $t$ is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t$=5.

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point is earned for using $B'(t)$

1 point is earned for the answer

Bob's acceleration is $B'(t)=3t^2-12t.B'(5)=3(25)-12(5)=15$ meters/min$^2$

| 0 | 1 | 2 |
|---|---|---|

The student response earns all of the following points:

1 point is earned for using $B'(t)$

1 point is earned for the answer

Bob's acceleration is $B'(t)=3t^2-12t$.

$B'(5)=3(25)-12(5)=15$ meters/min$^2$

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**7.** Use the data in the table to estimate the value of $v'(16)$ .

Please respond on separate paper, following directions from your teacher.

**Part A**

1 point is earned for the approximation

$$v'(16) \approx \frac{240-200}{20-12} = 5\ meters/min^2$$

| 0 | 1 |
| --- | --- |

The student response earns all of the following points:

1 point is earned for the approximation

$$v'(16) \approx \frac{240-200}{20-12} = 5\ meters/min^2$$

**8.** Based on the model $B$ from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

Please respond on separate paper, following directions from your teacher.

**Part D**

1 point is earned for the integral

1 point is earned for the antiderivative

1 point is earned for the answer

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$$\text{Avg vel} = \frac{1}{10}\int_0^{10}(t^3 - 6t^2 + 300)dt$$
$$= \frac{1}{10}\left[\frac{t^4}{4} - 2t^3 + 300t\right]_0^{10}$$
$$= \frac{1}{10}\left[\frac{10000}{4} - 2000 + 3000\right] = 350\,\text{meters/min}$$

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The student response earns all of the following points:

1 point is earned for the integral

1 point is earned for the antiderivative

1 point is earned for the answer


$$\text{Avg vel} = \frac{1}{10}\int_0^{10}(t^3 - 6t^2 + 300)dt$$
$$= \frac{1}{10}\left[\frac{t^4}{4} - 2t^3 + 300t\right]_0^{10}$$
$$= \frac{1}{10}\left[\frac{10000}{4} - 2000 + 3000\right] = 350\,\text{meters/min}$$


**9.** Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

Please respond on separate paper, following directions from your teacher.

### Part B

1 point is earned for the explanation

1 point is earned for the right Riemann sum

1 point is earned for the approximation

$\int_{40}^{0} |v(t)|\, dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \le t \le 40$ minutes.

$\int_{40}^{0} |v(t)|\, dt \approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)|$

$= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150$

$= 2400 + 1920 + 880 + 2400$

$= 7600$ meters

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The student response earns all of the following points:

1 point is earned for the explanation

1 point is earned for the right Riemann sum

1 point is earned for the approximation

$\int_{40}^{0} |v(t)|\, dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \le t \le 40$ minutes.

$\int_{40}^{0} |v(t)|\, dt \approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)|$

$= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150$

$= 2400 + 1920 + 880 + 2400$

$= 7600$ meters

A particle moves along the $x$-axis with position at time $t$ given by $x(t) = e^{-t} \sin t$ for $0 \le t \le 2\pi$.

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**10.** Find the time *t* at which the particle is farthest to the left. Justify your answer.

Please respond on separate paper, following directions from your teacher.

### Part A

2 points are earned for: $x'(t)$

1 point is earned for sets $x'(t) = 0$

1 point is earned for the answer

1 point is earned for the justification

$x'(t) = -e^{-t}\sin t + e^{-t}\cos t = -e^{-t}(\cos t - \sin t)$
$x'(t) = 0, when, \cos t = \sin t. Therefore, x'(t) = 0, on$
$0 \leq t \leq 2\pi, for, t = \frac{\pi}{4} and, t = \frac{5\pi}{4}$

The candidates for the absolute minimum are at

$t = 0, \frac{\pi}{4}, \frac{5\pi}{4} and, 2\pi.$

| $t$ | $x(t)$ |
|---|---|
| $0$ | $e^0 \sin(0) = 0$ |
| $\frac{\pi}{4}$ | $e^{-\frac{\pi}{4}} \sin\left(\frac{\pi}{4}\right) > 0$ |
| $\frac{5\pi}{4}$ | $e^{-\frac{5\pi}{4}} \sin\left(\frac{5\pi}{4}\right) < 0$ |
| $2\pi$ | $e^{-2\pi} \sin(2\pi) = 0$ |

The particle is farthest to the left when $t = \frac{5\pi}{4}$.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|

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The student response earns all of the following points:

2 points are earned for: $x'(t)$

1 point is earned for sets $x'(t) = 0$

1 point is earned for the answer

1 point is earned for the justification

$x'(t) = -e^{-t}\sin t + e^{-t}\cos t = -e^{-t}(\cos t - \sin t)$
$x'(t) = 0, when, \cos t = \sin t. Therefore, x'(t) = 0, on$
$0 \le t \le 2\pi, for, t = \frac{\pi}{4} and, t = \frac{5\pi}{4}$

The candidates for the absolute minimum are at

$t = 0, \frac{\pi}{4}, \frac{5\pi}{4} and, 2\pi.$

| $t$ | $x(t)$ |
|---|---|
| $0$ | $e^0 \sin(0) = 0$ |
| $\frac{\pi}{4}$ | $e^{-\frac{\pi}{4}} \sin\left(\frac{\pi}{4}\right) > 0$ |
| $\frac{5\pi}{4}$ | $e^{-\frac{5\pi}{4}} \sin\left(\frac{5\pi}{4}\right) < 0$ |
| $2\pi$ | $e^{-2\pi} \sin(2\pi) = 0$ |

The particle is farthest to the left when $t = \frac{5\pi}{4}$.

**11.** NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is

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given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function $f$ is assumed to be the set of all real numbers $x$ for which $f(x)$ is a real number.

| $t$ (hours) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $B(t)$ (miles per hour) | 1 | 8 | 1.5 | $-5$ | 11 |

Brandon and Chloe ride their bikes for $4$ hours along a flat, straight road. Brandon's velocity, in miles per hour, at time $t$ hours is given by a differentiable function $B$ for $0 \leq t \leq 4$. Values of $B(t)$ for selected times $t$ are given in the table above. Chloe's velocity, in miles per hour, at time $t$ hours is given by the piecewise function $C$ defined by

$$C(t) = \begin{cases} te^{4-t^2} & \text{for } 0 \leq t \leq 2 \\ 12 - 3t - t^2 & \text{for } 2 < t \leq 4. \end{cases}$$

(a) How many miles did Chloe travel from time $t = 0$ to time $t = 2$ ?

Please respond on separate paper, following directions from your teacher.

(b) At time $t = 3$, is Chloe's speed increasing or decreasing? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

(c) Is there a time $t$, for $0 \leq t \leq 4$, at which Brandon's acceleration is equal to $2.5$ miles per hour per hour? Justify your answer.

Please respond on separate paper, following directions from your teacher.

(d) Is there a time $t$, for $0 \leq t \leq 2$, at which Brandon's velocity is equal to Chloe's velocity? Justify your answer.

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Please respond on separate paper, following directions from your teacher.

### Part A

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

The first point is earned with a connection to the correct part of the piecewise defined function. $\int_0^2 C(t)\ dt$ by itself is not sufficient to earn the point.

The second point is earned with $-\frac{1}{2}e^{4-t^2}$ or with $-\frac{1}{2}e^u$ for $u = 4 - t^2$. Limits of integration are part of the third point.

The third point does not require units.

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The student response accurately includes **all three** of the criteria below.

- ☐ integral
- ☐ antiderivative
- ☐ answer

**Solution:**

$$\int_0^2 te^{4-t^2}\ dt = -\frac{1}{2}e^{4-t^2}\Big|_{t=0}^{t=2} = -\frac{1}{2} + \frac{1}{2}e^4$$

Chloe traveled $-\frac{1}{2} + \frac{1}{2}e^4$ miles from time $t = 0$ to time $t = 2$.

### Part B

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The first point is earned by showing both $C(3)$ and $C'(3)$ are less than zero or that both have the same sign.

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| 0 | 1 | 2 |
|---|---|---|

The student response accurately includes **both** of the criteria below.

- ☐ $C(3) < 0$ and $C'(3) < 0$
- ☐ answer with reason

**Solution:**

$C(3) = -6 < 0$

$C'(3) = -9 < 0$

Chloe's speed is increasing at time $t = 3$ because her velocity and acceleration have the same sign.

**Part C**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The second point is earned with the justification that $B$ is continuous and a numerical difference quotient equal to $2.5$. There is no requirement for the response to indicate that $B$ is differentiable, nor to refer to the Mean Value Theorem by name.

| 0 | 1 | 2 |
|---|---|---|

The student response accurately includes **both** of the criteria below.

- ☐ $\frac{B(4)-B(0)}{4-0}$
- ☐ answer with justification

**Solution:**

$B$ is differentiable $\Rightarrow B$ is continuous on $[0,\ 4]$.

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$\frac{B(4)-B(0)}{4-0} = \frac{11-1}{4-0} = 2.5$

By the Mean Value Theorem, there is a time $t$, for $0 < t < 4$, such that $B'(t) = 2.5$ miles per hour per hour.

**Part D**

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

The first point is earned by comparing $B$ and $C$, either by a subtraction statement or by comparing $B(t)$ and $C(t)$ for some $t$ in $\{0, 1, 2\}$.

The second point is earned with a justification that includes asserting the continuity of $B$ and $C$ or of $B - C$, and showing that $B(0) > C(0)$ and $B(2) < C(2)$. It is not required for the response to specifically refer to the Intermediate Value Theorem.

| 0 | 1 | 2 ✓ |
|---|---|---|

The student response accurately includes **both** of the criteria below.

- ☐ considers $B(t) - C(t)$
- ☐ answer with justification

**Solution:**

$B$ and $C$ are continuous on $[0, 2]$, therefore $B - C$ is continuous on $[0, 2]$.

$B(0) - C(0) = 1 - 0 > 0$

$B(2) - C(2) = 1.5 - 2 < 0$

By the Intermediate Value Theorem, there is a time $t$, for $0 < t < 2$, such that $B(t) - C(t) = 0$, or $B(t) = C(t)$.

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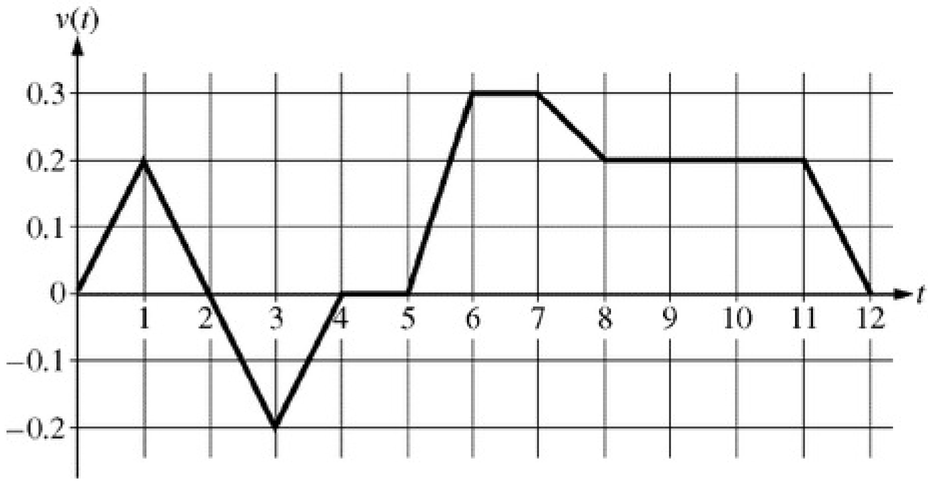


Caren rides her bicycle along a straight road from home to school, starting at home at time $t$=0 minutes and arriving at school at time $t$=12 minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$ , in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

**12.** Find the acceleration of Caren's bicycle at time $t$=7.5 minutes. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part A**

1 point is earned for answer

1 point is earned for unit

$$a(7.5) = v'(7.5) = \frac{v(8)-v(7)}{8-7} = -0.1 \text{ miles } / \text{minute}^2$$

| | | |
|---|---|---|
| 0 | 1 | 2 |

The student response earns two of the following points:

1 point is earned for answer

1 point is earned for unit

$$a(7.5) = v'(7.5) = \frac{v(8)-v(7)}{8-7} = -0.1 \text{ miles } / \text{minute}^2$$

**13.** Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point is earned for answer

1 point is earned for reason

Caren turns around to go back home at time t=2 minutes. This is the time at which her velocity changes from positive to negative.

| 0 | 1 | 2 |
|---|---|---|

The student response earns two of the following points:

1 point is earned for answer

1 point is earned for reason

Caren turns around to go back home at time t=2 minutes. This is the time at which her velocity changes from positive to negative.

**14.** Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t)=\pi/15\sin(\pi/12t)$ , where w(t) is in miles per minute for $0 \leq t \leq 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

**Part D**

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1 point is earned for integral

1 point is earned for value of larry's distance

1 point is earned for value of caren's distane and conclusion

$\int_0^{12} w(t)dt = 1.6$; Larry lives 1.6 miles fr om school.

$\int_0^{12} v(t)dt = 1.4$; Caren lives 1.4 miles fr om school.

Therefore, Caren lives closer to school.

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The students response earns three of the following points

1 point is earned for integral

1 point is earned for value of larry's distance

1 point is earned for value of caren's distane and conclusion

$\int_0^{12} w(t)dt = 1.6$; Larry lives 1.6 miles fr om school.

$\int_0^{12} v(t)dt = 1.4$; Caren lives 1.4 miles fr om school.

Therefore, Caren lives closer to school.

**15.** Using correct units, explain the meaning of $\int_0^{12} |v(t)|dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|dt$.

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Please respond on separate paper, following directions from your teacher.

**Part B**

1 point is earned for meaning of integral

1 point is earned for value of integral

$\int_0^{12} |v(t)|\, dt$is the total distance, in miles, that Caren rode during the 12 minutes fr om t=0 to t=12.

$$\int_0^{12} |v(t)|\, dt = \int_0^{2} v(t)dt - \int_4^{12} v(t)dt$$

=0.2+0.2+1.4=1.8 miles

| 0 | 1 | 2 |
|---|---|---|

The student response earns two of the following points:

1 point is earned for meaning of integral

1 point is earned for value of integral

$\int_0^{12} |v(t)|\, dt$ is the total distance, in miles, that Caren rode during the 12 minutes fr om t=0 to t=12.

$$\int_0^{12} |v(t)|\, dt = \int_0^{2} v(t)dt - \int_4^{12} v(t)dt$$

=0.2+0.2+1.4=1.8 miles

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| $t$ (hours) | 0 | 0.4 | 0.8 | 1.2 | 1.6 | 2.0 | 2.4 |
|---|---|---|---|---|---|---|---|
| $v(t)$ (miles per hour) | 0 | 11.8 | 9.5 | 17.2 | 16.3 | 16.8 | 20.1 |

Ruth rode her bicycle on a straight trail. She recorded her velocity $v(t)$ , in miles per hour, for selected values of $t$ over the interval $0 \le t \le 2.4$ , as shown in the table above, For $0 < t \le 2.4, v(t) > 0$ .

**16.** According to the model, $g(t) = \frac{24t+5\sin(6t)}{t+0.7}$, is Ruth's speed increasing or decreasing at time $t$=1.3 ? Give a reason reason for your answer.

Please respond on separate paper, following directions from your teacher.

**Part D**

The response can earn up to 2 points:

2 points: For correct conclusion with reason

$$\begin{aligned}\text{Velocity } &= g(1.3) = 18.096358 > 0\\ \text{Acceleration } &= g'(1.3) = 3.761152 > 0\end{aligned}$$

Ruth's speed is increasing at time $t = 1.3$ since velocity and acceleration have the same sign at this time.

| 0 | 1 | 2 |
|---|---|---|

The response earns both of the following points:

2 points: For correct conclusion with reason

$$\begin{aligned}\text{Velocity } &= g(1.3) = 18.096358 > 0\\ \text{Acceleration } &= g'(1.3) = 3.761152 > 0\end{aligned}$$

Ruth's speed is increasing at time $t = 1.3$ since velocity and acceleration have the same sign at this time.

**17.** Use the data in the table to approximate Ruth's acceleration at time $t=1.4$ hours. Show the computations that lead to your answer. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part A**

The response can earn up to 2 points:

1 point: For correct approximation

1 point: For correct units

$$a(1.4) \approx \frac{v(1.6) - v(1.2)}{1.6 - 1.2} = \frac{16.3 - 17.2}{1.6 - 1.2} = -2.25 \text{ miles/hr}^2$$

| 0 | 1 | 2 |
|---|---|---|

The response earns both of the following points:

1 point: For correct approximation

1 point: For correct units

$$a(1.4) \approx \frac{v(1.6) - v(1.2)}{1.6 - 1.2} = \frac{16.3 - 17.2}{1.6 - 1.2} = -2.25 \text{ miles/hr}^2$$

**18.** Using correct units, interpret the meaning of $\int_0^{2.4} v(t)\,dt$ in the context of the problem. Approximate $\int_0^{2.4} v(t)\,dt$ using a midpoint Riemann sum with three subintervals of equal length and values from the table.

Please respond on separate paper, following directions from your teacher.

**Part B**

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The response can earn up to 3 points:

1 point: For the correct interpretation

$\int_0^{2.4} v(t)dt$ is the total distance Ruth, traveled, in miles, from time $t = 0$ to time $t = 2.4$ hours.

1 point: For use of the midpoint Riemann sum

1 point: For the correct approximation

$$\int_0^{2.4} v(t)dt \approx (0.8)(11.8) + (0.8)(17.2) + (0.8)(16.8) = 36.64 \text{ miles}$$

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The response earns all three of the following points:

1 point: For the correct interpretation

$\int_0^{2.4} v(t)dt$ is the total distance Ruth, traveled, in miles, from time $t = 0$ to time $t = 2.4$hours.

1 point: For use of the midpoint Riemann sum

1 point: For the correct approximation

$$\int_0^{2.4} v(t)dt \approx (0.8)(11.8) + (0.8)(17.2) + (0.8)(16.8) = 36.64 \text{ miles}$$

**19.** For $0 \le t \le 4$ hours, Ruth's velocity can be modeled by the function $g$ given by $g(t) = \frac{24t+5\sin(6t)}{t+0.7}$. According to the model, what was Ruth's average velocity during the time interval $0 \le t \le 4$ ?

Please respond on separate paper, following directions from your teacher.

**Part C**

The response can earn up to 2 points:

## Week 5 Motion Review

1 point: For the correct integral

Average velocity = $\frac{1}{2.4}\int_0^{2.4} g(t)dt$

1 point: for the correct answer

Average velocity = 14.064 miles/hr

| 0 | 1 | 2 |
|---|---|---|

The response earns both of the following points:

1 point: For the correct integral

Average velocity = $\frac{1}{2.4}\int_0^{2.4} g(t)dt$

1 point: for the correct answer

Average velocity = 14.064 miles/hr

**20.** NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function $f$ is assumed to be the set of all real numbers $x$ for which $f(x)$ is a real number.

## Week 5 Motion Review

| $t$ (seconds) | 0 | 3 | 8 | 11 |
|---|---|---|---|---|
| $v(t)$ (meters per second) | $-15$ | $-12$ | $-8$ | $-3$ |

The velocity of a particle, $P$, moving along the $x$-axis is modeled by a differentiable function $v$, where time $t$ is measured in seconds and $v(t)$ is measured in meters per second. Selected values of $v(t)$ are shown in the table above.

(a) Use the data in the table to approximate $v'(10)$ using the average rate of change of $v(t)$ over the interval $8 \leq t \leq 11$. Show the computations that lead to your answer. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

(b) Interpret the meaning of $v'(10)$ in the context of the problem.

Please respond on separate paper, following directions from your teacher.

(c) Justify why there must be a time $t = k$, for $0 \leq k \leq 3$, when the velocity of the particle is $-13$ meters per second.

Please respond on separate paper, following directions from your teacher.

(d) Use a right Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{11} v(t)\,dt$. Show the computations that lead to your answer.

Please respond on separate paper, following directions from your teacher.

(e) Find $\int_4^{14} v'\left(\frac{t}{2}+1\right)dt$. Show the computations that lead to your answer.

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Please respond on separate paper, following directions from your teacher.

(f) Let $h(x) = \int_{3}^{\frac{11}{2}x} v(2t)\,dt$. Find $h'(1)$. Show the computations that lead to your answer.

Please respond on separate paper, following directions from your teacher.

(g) The position of a second particle, $Q$, can be modeled by a twice-differentiable function $g$. It is known that $g(2) = -5.5$, $g'(2) = 4$, and $g''(2) = 3$. Is the speed of particle $Q$ increasing or decreasing at time $t = 2$ ? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

(h) Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y^2 - xy$ with initial condition $f(3) = 2$. Write an equation for the line tangent to the graph of $f$ at the point $(3, 2)$.

Please respond on separate paper, following directions from your teacher.

**Part A**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

First point: Approximation.

Need to see both a difference and a quotient as the supporting work. Minimal answers that earn the point include $\frac{-3+8}{11-8}$, $\frac{-3-(-8)}{3}$, and $\frac{5}{11-8}$.

The answer does not need to be simplified, but any attempt at simplification must be correct.

Numerical values must be pulled from the table. $\frac{v(11)-v(8)}{11-8}$ alone will not earn the point as it does not pull values from the table (yet).

Second point: Units

$\frac{m}{s^2}$ end fraction or meters per second per second will earn the point.

## Week 5 Motion Review

The units must be tied to a number. Units alone do not earn this point.

Beware of $\left(\frac{m}{s}\right)^2$ or $\frac{m^2}{s}$ (which do not earn the point).

This point must be earned in part (a). If no units are present in part (a) and correct units are shown in part (b), this point is not earned.

| 0 | 1 | 2 ✓ |
|---|---|---|

The student response accurately includes **both** of the criteria below.

- ☐ Approximation
- ☐ Units

**Solution:**

$v'(10) \approx \frac{-3-(-8)}{11-8} = \frac{5}{3}$ meters per second per second

**Part B**

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

First point: Interpretation

There are two components to look for:

· "rate of change of velocity" or "acceleration" of the particle

· at $t = 10$

"The rate at which the velocity is increasing at $t = 10$" is acceptable. If the answer to part (a) was reported to be negative, "the rate at which the velocity is decreasing at $t = 10$" is acceptable. We will not penalize for the same error twice.

Units are not required in part (b). If units are presented for $t$, they must be $s$ (seconds).

| 0 | 1 ✓ |
|---|---|

## Week 5 Motion Review

The student response accurately includes the criteria below.

- ☐ Interpretation

**Solution:**

$v'(10)$ is the rate at which the velocity of the particle is changing, in meters per second per second, at time $t = 10$ seconds.

**Part C**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

First point: $v$ is continuous

It must be clear that the reason given as to why $v$ is continuous is that $v$ is differentiable.

Neither "$v$ is differentiable and continuous" nor "$v$ differentiable, $v$ continuous" are sufficient to earn this point.

"$v$ is differentiable hence continuous," by contrast, earns the point.

"$v$ differentiable $\therefore$ $v$ continuous" earns the point.

Second point: Bounding with table values

Must see the values $-15$, and $-12$ pulled from the table and correctly compared to $-13$, in words or using inequality symbols.

The Intermediate Value Theorem (IVT) does not need to be referenced by name.

That said, this point will not be earned if other theorems (MVT, etc.) are invoked.

Responses often "say too much"; especially common are responses that indicate that $v$ is always increasing on the interval $[0, 3]$. This point will not be earned if the response contains such errors in reasoning.

| 0 | 1 | 2 ✓ |
|---|---|---|

The student response accurately includes **both** of the criteria below.

- ☐ $v$ continuous
- ☐ Bounding $-13$ with table values

**Solution:**

$v$ is differentiable. $\Rightarrow v$ is continuous.

$v(0) = -15 < -13 < -12 = v(3)$

By the Intermediate Value Theorem, there must be a time $t = k$, for $0 \leq k \leq 3$, such that the velocity of the particle is $-13$ meters per second.

**Part D**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

First point: Form of right Riemann sum

Must see a sum of three terms, each of which is a product of two factors: $(b_1 \cdot h_1) + (b_2 \cdot h_2) + (b_3 \cdot h_3)$.

If exactly $5$ out of $6$ of these numbers $(b_1, b_2, b_3, h_1, h_2, h_3)$ are correctly placed in the above expression, we are convinced that the response carries the form of a right Riemann sum. In this case, award the first point. Such a response cannot earn the second point. If fewer than $5$ out of $6$ values are correct, no points are earned in this problem.

To earn this point, values do not need yet to be imported from the table: the expression $v(3) \cdot (3-0) + v(8) \cdot (8-3) + v(11) \cdot (11-8)$ is sufficient.

Once the first point is earned, it cannot be lost. Subsequent errors result in the second point not being earned.

second point: Answer

$-85$ or equivalent.

It is possible to earn this point without earning the first point. If there is credible evidence that an attempt was made to find a right Riemann sum but not enough work is shown, this point can be earned in the presence of the correct answer. For example, a sum of three terms with no evidence of products $(-36 + (-40) + (-9))$, or a graph with areas labeled and summed to $-85$ or equivalent, can earn this point.

The response of only $(-12) \cdot 3 + (-8) \cdot 5 + (-3) \cdot 3$ earns both points.

| 0 | 1 | 2 ✓ |
|---|---|---|

The student response accurately includes **both** of the criteria below.

☐ Form of right Riemann sum

## Week 5 Motion Review

☐ Answer

**Solution:**

$$\int_0^{11} v(t)\,dt \approx v(3)(3-0) + v(8)(8-3) + v(11)(11-8)$$

$$= -12 \cdot 3 + (-8) \cdot 5 + (-3) \cdot 3 = -85$$

**Part E**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

First point: Substitution

There are three elements to look for in a correct substitution. (another variable may be chosen instead of $u$.)

· $u = \frac{t}{2} + 1$

· $du = \frac{dt}{2}$ or equivalent

· Appropriate limits of integration throughout ($3$ and $8$ for $u$, $4$ and $14$ for $t$).

Substitution can be implicit. A coefficient of $2$ accompanying a correct antiderivative convinces us of this:

$$\int_4^{14} v'\left(\frac{t}{2} + 1\right) dt = 2v\left(\tfrac{t}{2} + 1\right)\Big|_4^{14}$$

To elaborate on the third element above: if a response shows an otherwise correct substitution but the limits of integration on a definite integral are unchanged ($4$ and $14$), this point is not earned.

Alternately, the integral can be worked with as an indefinite integral with the variable returned to $\frac{t}{2} + 1$ before evaluating as a means of eliminating the need to change limits. Such a response can earn the first point, if handled correctly.

Second point: Fundamental Theorem of Calculus

We must be convinced that the response has indicated an appropriate connection between $\int v'$ and $v$.

Expressions/equations that earn this point are:

· $k(v(8) - v(3))$ or $k\left(v\left(\frac{14}{2} + 1\right) - v\left(\frac{4}{2} + 1\right)\right)$, where $k$ is a nonzero constant.

· $kv\left(\frac{t}{2} + 1\right)\Big|_4^{14}$ or $k\,v(u)\Big|_3^8$, where $k$ is a nonzero constant.

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· $\int v'\left(\frac{t}{2}+1\right)dt = kv\left(\frac{t}{2}+1\right)$ or $k\int v'(u)\,du = kv(u)$, where $k$ is a nonzero constant, with or without appropriate limits of integration displayed on either integral.

· $\int_4^{14} v'\left(\frac{t}{2}+1\right)dt = k\,v\left(\frac{t^2}{4}+t\right)$ where $k$ is a nonzero constant is a common response. This does not earn the point. Likewise, similar mishandled antiderivatives do not earn the point.

Third point: Answer

8 or equivalent, with credible supporting work, earns this point.

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The student response accurately includes **all three** of the criteria below.

- ☐ Substitution
- ☐ Fundamental Theorem of Calculus
- ☐ Answer

**Solution:**

Let $u = \frac{t}{2}+1$. Then $du = \frac{dt}{2}$.

When $t = 4$, $u = 3$.

When $t = 14$, $u = 8$.

$$\int_4^{14} v'\left(\frac{t}{2}+1\right)dt = 2\int_3^8 v'(u)\,du$$

$$= 2\left(v(8) - v(3)\right)$$

$$= 2\left(-8 - (-12)\right) = 8$$

**Part F**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

First point: Fundamental Theorem of Calculus (FTC)

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Substitutes $\frac{11}{2}x$ into $v(2t)$. We must see either $v\left(2 \cdot \frac{11}{2}x\right)$ or $v(11x)$.

The expression $v\left(2 \cdot \frac{11}{2}x\right) - v\left(2\left(3\right)\right)$ or $v\left(2 \cdot \frac{11}{2}x\right) \cdot \frac{11}{2} - v\left(2\left(3\right)\right)$ will earn this point. The error of subtracting $v\left(2\left(3\right)\right)$ will be viewed as an error in applying the chain rule.

Beware of possible substitutions prior to using FTC:

· For example, $u = 2t$, $\frac{1}{2}du = dt$ and rewrites: $\int_3^{\frac{11}{2}x} v\left(2t\right)dt = \frac{1}{2}\int_6^{11x} v\left(u\right)du.$

· Or $u = \frac{11}{2}x$, $\frac{du}{dx} = \frac{11}{2}$ and rewrites: $\frac{d}{dx}\left[h\left(x\right)\right] = \frac{d}{du}\left[h\left(u\right)\right] \cdot \frac{du}{dx}$

· Such approaches can earn this point if done properly.

Second point: Chain rule

Multiplies by $\frac{11}{2}$

It is not possible to earn this point without having earned the first point.

Depending on substitutions, this may look more like $\frac{1}{2}$ and $11$ multiplied separately.

Third point: Answer

$-\frac{33}{2}$ or equivalent

It is not possible to earn this point without having earned both the first and second points.

| 0 | 1 | 2 | 3 ✓ |
|---|---|---|---|

The student response accurately includes **all three** of the criteria below.

- ☐ Fundamental Theorem of Calculus
- ☐ Chain rule
- ☐ Answer

**Solution:**

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$$h'(x) = \frac{d}{dx}\int_3^{\frac{11}{2}x} v(2t)\ dt = v\left(2 \cdot \frac{11}{2}x\right) \cdot \frac{11}{2}$$

$$h'(1) = v(11) \cdot \frac{11}{2} = -3 \cdot \frac{11}{2} = -\frac{33}{2}$$

**Part G**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

First point: Narrows consideration to $g'$ and $g''$

You must be convinced the response considers only the functions $g'$ and $g''$ as being critical to the reasoning.

Second point: Answer with reason

A correct response must:

· Conclude "increasing"

· Appeal to the signs of $g'$ and $g''$

It is not possible to earn this point without having earned the first point.

A response that mentions velocity and acceleration must tie velocity to $g'$ and acceleration to $g''$. A response that says only "increasing because velocity and acceleration are both negative" earns neither of the two points. It is not clear that the response is considering $g'$ and $g''$. A response that calls out all three given values ($g(2)$, $g'(2)$, and $g''(2)$) initially might still earn one or both points. If the response goes on to focus upon/reason from the values of $g'$ and $g''$ only, both points can be earned.

The phrase "speeding up" will be interpreted as "increasing."

| 0 | 1 | 2 ✓ |
|---|---|---|

The student response accurately includes **both** of the criteria below.

- ☐ Narrows consideration to $g'$ and $g''$
- ☐ Answer with reason

**Solution:**

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Because $g'(2) = 4 > 0$, the velocity of the particle is positive. Because $g''(2) = 3 > 0$, the acceleration of the particle is positive. Therefore, because both the velocity and acceleration are positive, the speed of the particle is increasing at time $t = 2$.

**Part H**

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

First point: Tangent line equation

This point is earned only for the correct tangent line equation written in any form.

Some common forms:

- $y - 2 = \left((2)^2 - 3 \cdot 2\right)(x - 3)$
- $y - 2 = -2(x - 3)$
- $y = -2x + 8$

$y - 2 = \left(y^2 - xy\right)(x - 3)$ does not earn the point. This is not the equation of a line. The response can go on to present one of the correct linear forms, and then would earn the point.

$f(x) = -2(x - 3) + 2$ or equivalent does not earn the point. $f(x)$ was defined in this problem as the particular solution to the differential equation, so it cannot also be a tangent line.

Some responses have the correct tangent line equation written initially, with subsequent errors in simplification. Such responses do not earn the point.

The point can be earned after pursuing a less fruitful approach (such as solving the differential equation) and then restarting.

| 0 | 1 ✓ |
|---|---|

The student response accurately includes the criteria below.

☐ Tangent line equation

**Solution:**

$$\left.\frac{dy}{dx}\right|_{(3,\,2)} = 2^2 - 3 \cdot 2 = -2$$

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$y=-2(x-3)+2$