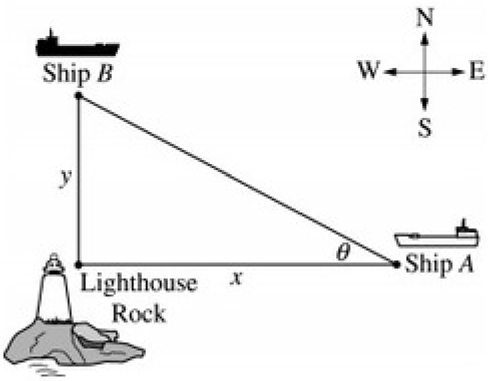
# Week 6 Related Rates




Ship $A$ is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship $B$ is traveling due north away from Lighthouse rock at a speed of 10 km/hr. Let $x$ be the distance between Ship $A$ and Lighthouse Rock at time $t$, and let $y$ be the distance between Ship $B$ and Lighthouse Rock at time $t$, as shown in the figure above.

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**1.** Let $\theta$ be the angle shown in the figure. Find the rate of change of $\theta$, radians per hour, when $x = 4$ km and $y = 3$ km.

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point is earned for the expression for $\theta$ in terms of x and y

2 points are earned for differentiation with respect to $t$ < -2 > chain rule, quotient rule, or transcendental function error

1 point is earned for the evaluation

$$\tan\theta = \frac{y}{x}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$$

At x = 4 and y = 3, $\sec\theta = \frac{5}{4}$

$$\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4)-(-15)(3)}{16}\right)$$

$$= \frac{85}{25} = \frac{17}{5} radians/hr$$

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| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 ✓ |

The student response earns all of the following points:

1 point is earned for the expression for $\theta$ in terms of x and y

2 points are earned for differentiation with respect to *t* < -2 > chain rule, quotient rule, or transcendental function error

1 point is earned for the evaluation

$$\tan\theta = \frac{y}{x}$$
$$\sec^2\theta\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$$

At x = 4 and y = 3, $\sec\theta = \frac{5}{4}$

$$\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4)-(-15)(3)}{16}\right)$$
$$= \frac{85}{25} = \frac{17}{5} radians/hr$$

**2.** Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

Please respond on separate paper, following directions from your teacher.

**Part B**

1 point is earned for the expression for distance

2 points are earned for the differentiation with respect to *t*

< -2 > chain rule error

1 point is earned for evaluation

$$r^2 = x^2 + y^2$$
$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

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or explicitly:

$$r=\sqrt{x^2+y^2}$$
$$\frac{dr}{dt}=\frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right)$$

At x = 4, y = 3,

$$\frac{dr}{dt}=\frac{4(-15)+3(10)}{5}=-6km/hr$$

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|

The student response earns all of the following points:

1 point is earned for the expression for distance

2 points are earned for the differentiation with respect to *t*

< -2 > chain rule error

1 point is earned for evaluation

$$r^2=x^2+y^2$$
$$2r\frac{dr}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$$

or explicitly:

$$r=\sqrt{x^2+y^2}$$
$$\frac{dr}{dt}=\frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right)$$

At x = 4, y = 3,

$$\frac{dr}{dt}=\frac{4(-15)+3(10)}{5}=-6km/hr$$

**3.** NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label

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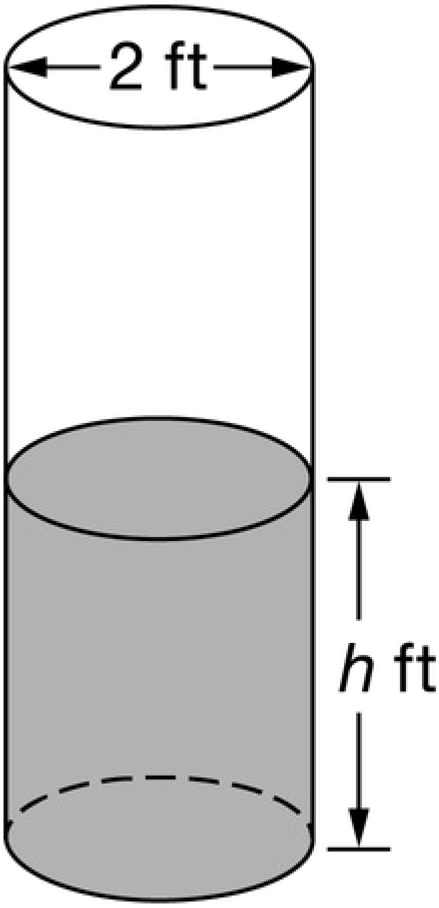
any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function $f$ is assumed to be the set of all real numbers $x$ for which $f(x)$ is a real number.




A cylindrical barrel with a diameter of $2$ feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height $h$ of the water in the barrel with respect to time $t$ is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where $h$ is measured in feet and $t$ is measured in seconds. (The volume $V$ of a cylinder with radius $r$ and height $h$ is $V = \pi r^2 h$.)

a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is $4$ feet. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

b) When the height of the water is $3$ feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

Please respond on separate paper, following directions from your teacher.

c) At time $t = 0$ seconds, the height of the water is $5$ feet. Use separation of variables to find an expression for $h$ in terms of $t$.

Please respond on separate paper, following directions from your teacher.

**Part A**

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

The first point is earned for the derivative of the volume equation with respect to time. Responses that differentiate $V = \pi r^2 h$ correctly using the product rule also earn the first point.

| 0 | 1 | 2 ✓ |
|---|---|---|

The student response accurately includes **both** of the criteria below.

- ☐ $\frac{dV}{dt} = \pi \frac{dh}{dt}$
- ☐ answer with units

**Solution:**

$$V = \pi r^2 h = \pi(1)^2 h = \pi h$$

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$\frac{dV}{dt}\Big|_{h=4} = \pi\frac{dh}{dt}\Big|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5}$ cubic feet per second

**Part B**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The first point is earned for the derivative of $-\frac{1}{10}\sqrt{h}$.

The second point is earned for $\frac{d^2h}{dt^2}$. Note that a response of $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt}$ earns both the first and second points.

| 0 | 1 | 2 | 3 ✓ |
|---|---|---|---|

The student response accurately includes **all three** of the criteria below.

- ☐ $\frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}}$
- ☐ $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt}$
- ☐ answer with explanation

**Solution:**

$$\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$$

Because $\frac{d^2h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is **3** feet.

**Part C**

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

Zero out of 4 points are earned if no separation of variables.

At most 2 out of 4 points can be earned [1-1-0-0] if there is no constant of integration.

Both antiderivatives must be correct to earn the second point.

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In order to earn the third point, at least one of the antiderivatives must be correct, and the $h$ must be in a function that is not logarithmic or linear. In addition, the constant of integration must be included immediately following the antiderivatives.

The fourth point requires an expression for $h\left(t\right)$.

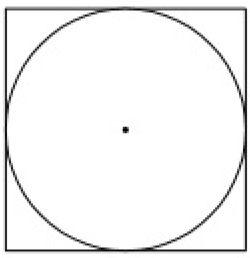
| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|

The student response accurately includes **all four** of the criteria below.

- ☐ separation of variables
- ☐ antiderivatives
- ☐ constant of integration and uses initial condition
- ☐ $h\left(t\right)$

**Solution:**

$$\frac{dh}{\sqrt{h}} = -\frac{1}{10}\,dt$$
$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$$
$$2\sqrt{h} = -\frac{1}{10}t + C$$
$$2\sqrt{5} = -\frac{1}{10}\cdot 0 + C \;\Rightarrow\; C = 2\sqrt{5}$$
$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$
$$h\left(t\right) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$$



A circle is inscribed in a square as shown in the figure above. The circumference of the circle is increasing at a

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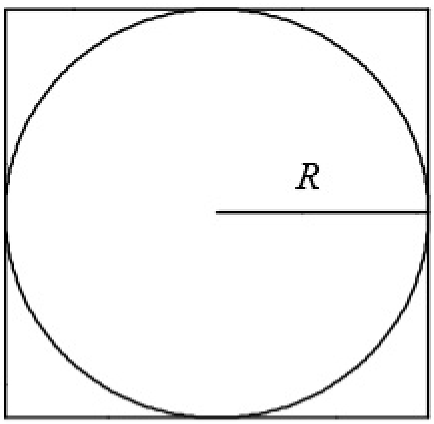
constant rate of 6 inches per second. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius $r$ has circumference $C=2\pi r$ and area $A=\pi r^2$ )

**4.** Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part A**

1 point is earned for correctly expressing the perimeter of square in terms of circle data




1 point is earned for correctly differentiating to find $\frac{dP}{dt}$

$P = 8R$
$\frac{dP}{dt} = 8\frac{dR}{dt}$

1 point is earned for the correct answer $\frac{dC}{dt} = 6$

$6 = \frac{dC}{dt} = 2\pi\frac{dR}{dt}$

1 point is earned for correctly solving for $\frac{dP}{dt}$

$\frac{dR}{dt} = \frac{3}{\pi}; \frac{dP}{dt} = \frac{24}{\pi}$ inches/second

≈ 7.639 inches/second

1 point is earned for correct units for answers in BOTH (a) and (b)

Units: inches/second in (a)

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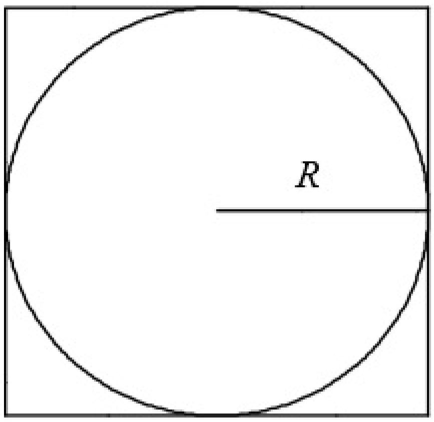
square inches/second in (b)

0/1 if no use of $\frac{dC}{dt}$

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|

The student response earns four of the following points:

1 point is earned for correctly expressing the perimeter of square in terms of circle data




1 point is earned for correctly differentiating to find $\frac{dP}{dt}$

$$P = 8R$$
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$\frac{dR}{dt} = \frac{3}{\pi}; \frac{dP}{dt} = \frac{24}{\pi}$ inches/second

≈ 7.639 inches/second

1 point is earned for correct units for answers in BOTH (a) and (b)

Units: inches/second in (a)

square inches/second in (b)

0/1 if no use of $\frac{dC}{dt}$

**5.** At the instant when the area of the circle is $25\pi$ square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part B**

1 point is earned for the correct area of region between the square and circle

$Area\ =\ 4R^2\ -\pi R^2$

1 point is earned for the correct derivative of area with respect to $t$

1/1: for derivatives of areas of square and circle separately

0/1: if area of either square or circle is linear

$$\frac{d(\text{Area})}{dt} = 8R\frac{dR}{dt} - 2\pi R\frac{dR}{dt}$$
$$= (4-\pi)2R\frac{dR}{dt}$$

1 point is earned for correctly using $R = 5$ in derivative

Area of circle = $25\pi =\ \pi R^2\ \ R = 5$

1 point is earned for the correct answer

$$\frac{d(\text{Area})}{dt} = \frac{120}{\pi} - 30\ \text{inches}^2/\sec ond$$
$$= (4-\pi)\ \frac{30}{\pi}\ \text{inches}^2/\sec ond$$
$$\approx 8.197\ \text{inches}^2/\sec ond$$

1 point is earned for correct units for answers in both (a) and (b)

Units: inches/second in (a)

square inches/second in (b)

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| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|

The student response earns five of the following points:

1 point is earned for the correct area of region between the square and circle

$Area = 4R^2 - \pi R^2$

1 point is earned for the correct derivative of area with respect to *t*

1/1: for derivatives of areas of square and circle separately

0/1: if area of either square or circle is linear

$$\frac{d(\text{Area})}{dt} = 8R\frac{dR}{dt} - 2\pi R\frac{dR}{dt}$$
$$= (4-\pi)2R\frac{dR}{dt}$$

1 point is earned for correctly using $R = 5$ in derivative

Area of circle = $25\pi = \pi R^2 \quad R = 5$

1 point is earned for the correct answer

$$\frac{d(\text{Area})}{dt} = \frac{120}{\pi} - 30\ \text{inches}^2/\sec ond$$
$$= (4-\pi)\ \frac{30}{\pi}\ \text{inches}^2/\sec ond$$
$$\approx 8.197\ \text{inches}^2/\sec ond$$

1 point is earned for correct units for answers in both (a) and (b)

Units: inches/second in (a)

square inches/second in (b)

The length of a solid cylindrical cord of elastic material is 32 inches. A circular cross section of the cord has radius 1/2 inch.

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**6.** The cord is stretched lengthwise at a constant rate of 18 inches per minute.Assuming that the cord maintains a cylindrical shape and a constant volume, at what rate is the radius of the cord changing one

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minute after the stretching begins? Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part B**

1 point is earned for $h = 50$

1 point is earned for $\frac{dv}{dt} = 0$

2 points are earned for differentiation

1 points i earned for $r = \frac{2}{5}$

1 points i earned for answer

$0 = \frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$;
$t = 1, h = 50$ and
so $8\pi{=}\pi\mathrm{r}^2 \cdot 50$,
so $r = \frac{2}{5}$
Therefore $0 = 2\pi(\frac{2}{5})(50)\frac{dr}{dt} + \pi\left(\frac{2}{5}\right)^2(18)$
$= \pi(40\frac{dr}{dt} + \frac{72}{25})$
$\frac{dr}{dt} = -\frac{9}{125}\mathrm{in/\ min}$

or

$V = 8\pi = \pi r^2 h,$ so $r = \sqrt{\frac{8}{h}}$
Therefore $\frac{dr}{dt} = \frac{1}{2}\left(\frac{8}{h}\right)^{-\frac{1}{2}} \cdot \left(\frac{-8}{h^2}\right) \cdot \left(\frac{dh}{dt}\right)$
$at$ t $=1$, $h = 50$ so
$\frac{dr}{dt} = \frac{1}{2}\left(\frac{8}{50}\right)^{-\frac{1}{2}}\left(\frac{-8}{2500}\right) . (18)$
$= -\frac{9}{125}\mathrm{in/min}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|

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The student response earns six of the following points:

1 point is earned for $h = 50$

1 point is earned for $\frac{dv}{dt} = 0$

2 points are earned for differentiation

1 points i earned for $r = \frac{2}{5}$

1 points i earned for answer

$0 = \frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$;
$t = 1, h = 50$ and
so $8\pi = \pi r^2 \cdot 50$,
so $r = \frac{2}{5}$

Therefore $0 = 2\pi(\frac{2}{5})(50)\frac{dr}{dt} + \pi\left(\frac{2}{5}\right)^2(18)$
$= \pi(40\frac{dr}{dt} + \frac{72}{25})$
$\frac{dr}{dt} = -\frac{9}{125}$in/min

or

$V = 8\pi = \pi r^2 h$, so $r = \sqrt{\frac{8}{h}}$

Therefore $\frac{dr}{dt} = \frac{1}{2}\left(\frac{8}{h}\right)^{-\frac{1}{2}} \cdot \left(\frac{-8}{h^2}\right) \cdot \left(\frac{dh}{dt}\right)$
$at$ t $=1$, $h = 50$ so
$\frac{dr}{dt} = \frac{1}{2}\left(\frac{8}{50}\right)^{-\frac{1}{2}}\left(\frac{-8}{2500}\right) . (18)$
$= -\frac{9}{125}$in/min

| $t$ (days) | 0 | 10 | 22 | 30 |
|---|---|---|---|---|
| $W'(t)$ (GL per day) | 0.6 | 0.7 | 1.0 | 0.5 |

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The twice-differentiable function $W$ models the volume of water in a reservoir at time $t$, where $W(t)$ is measured in gigaliters (GL) and $t$ is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \leq t \leq 30$ days. At time $t$ =30, the reservoir contains 125 gigaliters of water.

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**7.** The equation $A = 0.3W^{2/3}$ gives the relationship between the area $A$, in square kilometers, of the surface of the reservoir, and the volume of water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of $A$, in square kilometers per day, with respect to $t$ when $t = 30$ days.

Please respond on separate paper, following directions from your teacher.

**Part D**

Two points are earned for $\frac{dA}{dt}$

One point is earned for the answer

$$\frac{dA}{dt} = (0.3)\,\frac{2}{3}W^{-1/3}\cdot\frac{dW}{dt} = \frac{0.2}{\sqrt[3]{W}}\cdot\frac{dW}{dt}$$
$$\left.\frac{dA}{dt}\right|_{t=30} = \frac{0.2}{\sqrt[3]{125}}\cdot 0.5 = 0.02$$

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The student earns all of the following points:

Two points are earned for $\frac{dA}{dt}$

One point is earned for the answer

$$\frac{dA}{dt} = (0.3)\,\frac{2}{3}W^{-1/3}\cdot\frac{dW}{dt} = \frac{0.2}{\sqrt[3]{W}}\cdot\frac{dW}{dt}$$
$$\left.\frac{dA}{dt}\right|_{t=30} = \frac{0.2}{\sqrt[3]{125}}\cdot 0.5 = 0.02$$

A tight rope is stretched 30 feet above the ground between the Jay and the Tee buildings, which are 50 feet apart. A tightrope walker, walking at a constant rate of 2 feet per second from point $A$ to point $B$, is illuminated by a spotlight 70 feet above point $A$, as shown in the diagram.

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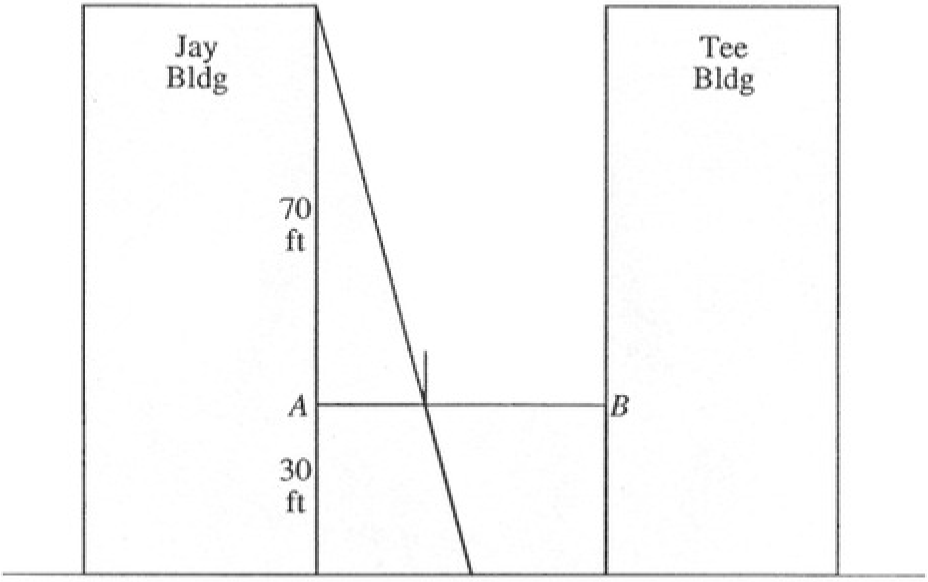



**8.** How fast is the shadow of the tightrope walker's feet moving along the ground when she is midway between the buildings? (Indicate units of measure.)

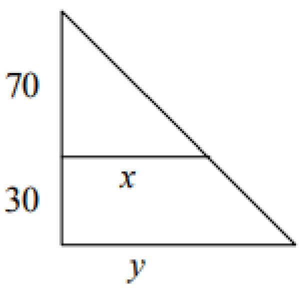
Please respond on separate paper, following directions from your teacher.

### Part A

1 point is earned for establishing two-variable equation relating variables "x" and "y"

1 point is earned for establishing equation relating rates dx dt , dy dt

1 point is earned for finding answer




$\frac{y}{100} = \frac{x}{70}$

## Week 6 Related Rates

$$70\,\frac{dy}{dt} = 100\,(2)$$
$$\frac{dy}{dt} = \frac{10}{7}(2)\ \frac{20}{7}\ \text{ft/sec}$$

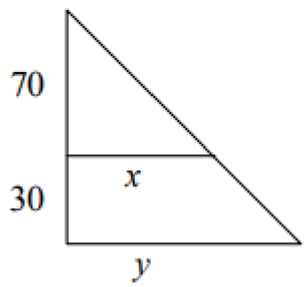
| 0 | 1 | 2 | 3 |
|---|---|---|---|

The student response earns three of the following points:

1 point is earned for establishing two-variable equation relating variables "x" and "y"

1 point is earned for establishing equation relating rates dx dt , dy dt

1 point is earned for finding answer




$$\frac{y}{100} = \frac{x}{70}$$

$$70\,\frac{dy}{dt} = 100\,(2)$$
$$\frac{dy}{dt} = \frac{10}{7}(2)\ \frac{20}{7}\ \text{ft/sec}$$

**9.** How fast is the shadow of the tightrope walker's feet moving up the wall of the Tee Building when she is 10 feet from point *B*? (Indicate units of measure.)

Please respond on separate paper, following directions from your teacher.

**Part C**

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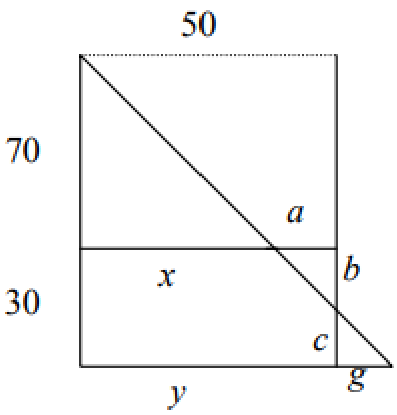
1 point is earned for establishing proportion relating horizontal variables $x$,$a$,or $g$ to vertical variables $b$ or $c$

2 points are earned for establishing equation relating rates from student's proportion

Max 1/2 if student stops before getting an equation involving at most two variables and their rates

1 point is earned for answer (including units)




$$\frac{y}{100} = \frac{x}{70}$$

$$70\frac{dy}{dt} = 100\frac{dx}{dt}\ OR\ \frac{dy/dt}{100} = \frac{dx/dt}{70}$$

$$\frac{b}{a} = \frac{70}{x}$$

$$\frac{b}{50-x} = \frac{70}{x}$$

$$bx = 3500 - 70x$$

$$b\frac{bx}{dt} + x\frac{db}{dt} = -70\frac{dx}{dt}$$

$$\left(\frac{35}{2}\right)(2) + (40)\frac{db}{dt} = -70\,(2)$$

$$\frac{db}{dt} = -\frac{35}{8}$$

$$\frac{35}{8}\text{ft}/\sec$$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |

The student response earns four of the following points:

1 point is earned for establishing proportion relating horizontal variables $x$,$a$,or $g$ to vertical variables $b$ or $c$

2 points are earned for establishing equation relating rates from student's proportion

Max 1/2 if student stops before getting an equation involving at most two variables and their rates

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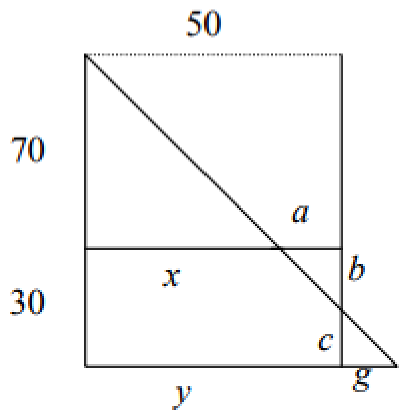
1 point is earned for answer (including units)




$\frac{y}{100} = \frac{x}{70}$

$70\frac{dy}{dt} = 100\frac{dx}{dt}\ OR\ \frac{dy/dt}{100} = \frac{dx/dt}{70}$

$\frac{b}{a} = \frac{70}{x}$

$\frac{b}{50-x} = \frac{70}{x}$

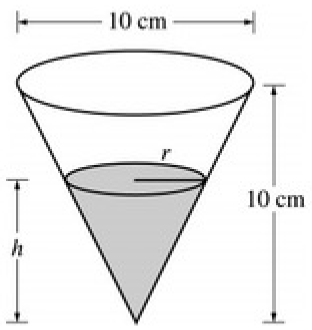
$bx = 3500 - 70x$

$b\frac{bx}{dt} + x\frac{db}{dt} = -70\frac{dx}{dt}$

$\left(\frac{35}{2}\right)(2) + (40)\frac{db}{dt} = -70(2)$

$\frac{db}{dt} = -\frac{35}{8}$

$\frac{35}{8}\text{ft}/\text{ sec}$




A container has the shape of an open right circular cone, as shown in the figure above. The height of the contained is 10 cm and the diameter of the opening is 10 cm. Water in the contained is evaporating so that its depth *h* is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

**10.** Show that the rate of change of the volume of water in the contained due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point is earned for shows $\frac{dV}{dt}$= k area

1 point is earned for identifies constant of proportionality

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt} = -\frac{3}{40}\pi h^2$$
$$= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10} \cdot area$$

The constant of proportionality is $-\frac{3}{10}$.

| 0 | 1 | 2 |
|---|---|---|

The student response earns all of the following points:

1 point is earned for shows $\frac{dV}{dt}$= k area

1 point is earned for identifies constant of proportionality

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt} = -\frac{3}{40}\pi h^2$$
$$= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10} \cdot area$$

The constant of proportionality is $-\frac{3}{10}$.

**11.** Find the rate of change of the volume of water in the contained, with respect to time, when $h$ = 5 cm. Indicate units of measurement.

## Week 6 Related Rates

Please respond on separate paper, following directions from your teacher.

**Part B**

1 point is earned for $r = \frac{1}{2}h, in(a)or(b)$

1 point is earned for V as a function of one variable in (a) or (b)

or $\frac{dr}{dt}$

2 points are earned for $\frac{dV}{dt}$

<-2> chain rule or product rule error

1 point is earned for the evaluation at $h = 5$

$$\frac{r}{h} = \frac{5}{10}, so, r = \frac{1}{2}h$$
$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3; \frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$$
$$\frac{dV}{dt}\Big|_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi, \ {cm^3}/{hr}$$
$$OR$$
$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right); \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$
$$\frac{dV}{dt}\Big|_{h=5,r=\frac{5}{2}} = \frac{1}{3}\pi\left(\left|\left(\frac{25}{5}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\right)$$
$$= -\frac{15}{8}\pi, \ {cm^3}/{hr}$$

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|

The student response earns all of the following points:

1 point is earned for $r = \frac{1}{2}h, in(a)or(b)$

1 point is earned for V as a function of one variable in (a) or (b)

or $\frac{dr}{dt}$

## Week 6 Related Rates

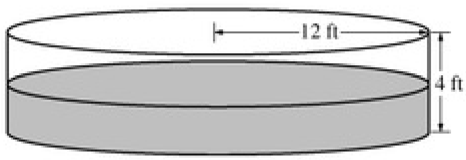
2 points are earned for $\frac{dV}{dt}$

<-2> chain rule or product rule error

1 point is earned for the evaluation at $h = 5$

$$\frac{r}{h} = \frac{5}{10}, so, r = \frac{1}{2}h$$
$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3; \frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$$
$$\left.\frac{dV}{dt}\right|_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi,\ {cm^3}/{hr}$$
$$OR$$
$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right); \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$
$$\left.\frac{dV}{dt}\right|_{h=5, r=\frac{5}{2}} = \frac{1}{3}\pi\Big(\Big|\left(\frac{25}{5}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\Big)$$
$$= -\frac{15}{8}\pi,\ {cm^3}/{hr}$$

| $t$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $P(t)$ | 0 | 46 | 53 | 57 | 60 | 62 | 63 |




The figure above shows an above ground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t$=0. During the time interval $0 \leq t \leq 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of $t$. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t)=25e^{-0.05t}$.

(Note: The volume $V$ of a cylinder with radius $r$ and height $h$ is given by $V=\pi r^2 h$.)

---

**12.** Find the rate at which the volume of water in the pool is increasing at time $t$=8 hours. How fast is the water level in the pool rising at $t$=8 hours? Indicate units of measure in both answers.

Please respond on separate paper, following directions from your teacher.

**Part D**

1 point is earned for $V'(8)$

## Week 6 Related Rates

1 point is earned for equation relating dv dt and dh dt

1 point is earned for dh dt | t=8

1 point is earned for units of ft3/ hr and ft/ hr

$V'(t)$=$P(t)$-$R(t)$

$V'(8) = P(8) - R(8) = 60 - 25e^{-0.4} = 43.241 \text{ or } 43.242 \text{ ft}^3/\text{hr}$

$\text{V} = \pi(12)^2 h$

dV dt =144π

dh dt dh dt | t=8 = 1 144π • dV dt | t=8 = 0.095 or 0.096 ft/hr ft/

| 0 | 1 | 2 | 3 | 4 |
| --- | --- | --- | --- | --- |

The student response earns all of the following points:

1 point is earned for $V'(8)$

1 point is earned for equation relating dv dt and dh dt

1 point is earned for dh dt | t=8

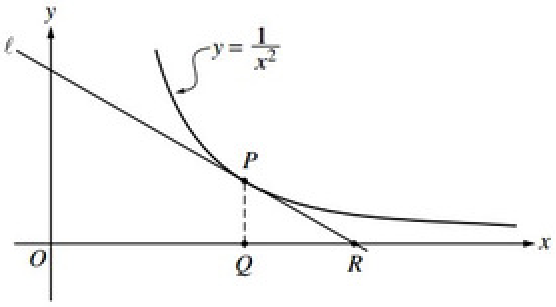
1 point is earned for units of ft3/ hr and ft/ hr

$V'(t)$=$P(t)$-$R(t)$

$\text{V} = \pi(12)^2 h$

$\frac{dh}{dt} = \Big|_{t=8} = \frac{1}{144\pi} \cdot \frac{dv}{dt}\Big|_{t=8} = 0.095 \text{ or } 0.096 \text{ ft/hr}$

## Week 6 Related Rates




In the figure above, line $L$ is tangent to the graph of $y = \frac{1}{x^2}$ at point $P$, with coordinates $\left(w, \frac{1}{w^2}\right)$, where $w > 0$. Point $Q$ has coordinates $(w, 0)$. Line $L$ crosses the x-axis at point $R$, with coordinates $(k, 0)$.

**13.** Suppose that $w$ is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of $k$ with respect to time?

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point is earned for the answer using: $\frac{dw}{dt} = 7$

$$\frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt} = \frac{3}{2} \cdot 7 = \frac{21}{2};\ \frac{dk}{dt}\Big|_{w=5} = \frac{21}{2}$$

| 0 | 1 |
|---|---|

The student response earns all of the following points:

1 point is earned for the answer using: $\frac{dw}{dt} = 7$

$$\frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt} = \frac{3}{2} \cdot 7 = \frac{21}{2};\ \frac{dk}{dt}\Big|_{w=5} = \frac{21}{2}$$

## Week 6 Related Rates

**14.** Suppose that $w$ is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of the area of n$PQR$ with respect to time? Determine whether the area is increasing or decreasing at this instant.

Please respond on separate paper, following directions from your teacher.

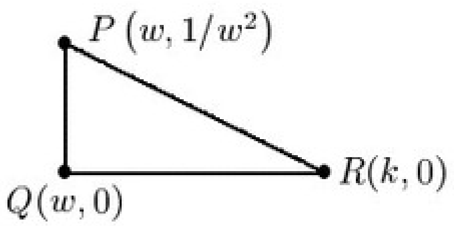
**Part D**

1 point is earned for the area in term or $w$ and/or $k$

1 point is earned for: $\frac{dA}{dt}$ implicitly

1 point is earned for: $\left.\frac{dA}{dt}\right|_{w=5} u \sin g \frac{dw}{dt} - 7$

1 point is earned for the conclusion




$$A = \frac{1}{2}(k-w)\frac{1}{w^2} = \frac{1}{2}\left(\frac{3}{2}w - w\right)\frac{1}{w^2} = \frac{1}{4w}$$
$$\frac{dA}{dt} = -\frac{1}{4w^2}\frac{dw}{dt}$$
$$\left.\frac{dA}{dt}\right|_{w=5} = -\frac{1}{100} \cdot 7 = -0.07$$

Therefore, area is decreasing.

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |

The student response earns all of the following points:

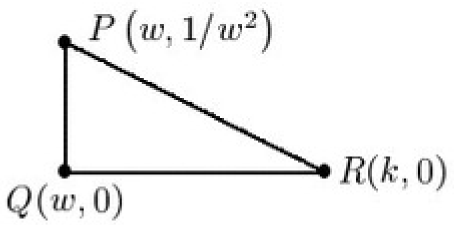
1 point is earned for the area in term or $w$ and/or $k$

1 point is earned for: $\frac{dA}{dt}$ implicitly

## Week 6 Related Rates

1 point is earned for: $\left.\frac{dA}{dt}\right|_{w=5} u \sin g \frac{dw}{dt} - 7$

1 point is earned for the conclusion




$A = \frac{1}{2}(k - w)\frac{1}{w^2} = \frac{1}{2}\left(\frac{3}{2}w - w\right)\frac{1}{w^2} = \frac{1}{4w}$
$\frac{dA}{dt} = -\frac{1}{4w^2}\frac{dw}{dt}$
$\left.\frac{dA}{dt}\right|_{w=5} = -\frac{1}{100} \cdot 7 = -0.07$

Therefore, area is decreasing.

A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. The can is initially empty, and rain enters the can during a 60-day period. The height of water in the can is modeled by the function $S$, where $S(t)$ is measured in millimeters and $t$ is measured in days for $0 \le t \le 60$. The rate at which the height of the water is rising in the can is given by $S'(t)=2\sin(0.03t)+1.5$.

---

**15.** Assuming no evaporation occurs, at what rate is the volume of water in the can changing at time $t$=7? Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point is earned for relationship between $V$ and $S$

1 point is earned for the answer

1 point earned for units in (b) or (c).

## Week 6 Related Rates

$V(t) = 100\,\pi\,\mathrm{S}(t)$
$V'(7) = 100\,\pi\,\mathrm{S}'(7) = 602.218$

The volume of water in the can is increasing at a rate of $602.218\ \mathrm{mm}^3/day$.

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The student response all of the following points:

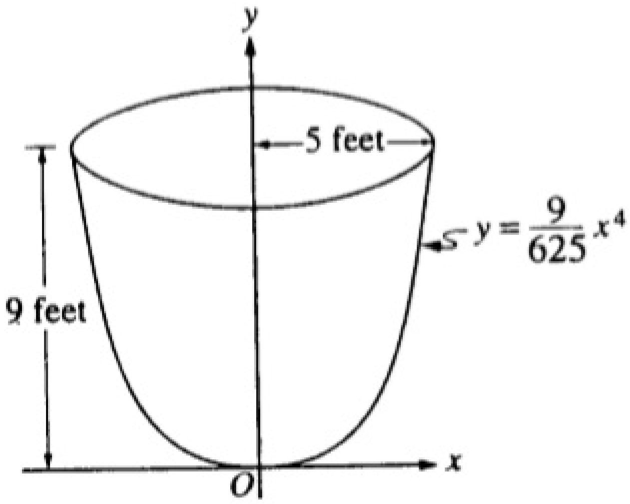
1 point is earned for relationship between $V$ and $S$

1 point is earned for the answer

1 point earned for units in (b) or (c).

$V(t) = 100\,\pi\,\mathrm{S}(t)$
$V'(7) = 100\,\pi\,\mathrm{S}'(7) = 602.218$

The volume of water in the can is increasing at a rate of $602.218\ \mathrm{mm}^3/day$.




An oil storage tank has the shape as shown above, obtained by revolving the curve $y = \frac{9}{625}x^4$ from x=0 to x=5 about the $y$-axis, where $x$ and $y$ are measured in feet.

## Week 6 Related Rates

Oil flows into the tank at the constant rate of 8 cubic feet per minute.

---

**16.** Let *h* be the depth, in feet, of oil in the tank. How fast is the depth of oil in the tank increasing when *h* = 4 ? Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point is earned for the correct volume as definite integral using *h*

Finding $\frac{dV}{dt}$

$$V = \pi \int_0^h \frac{25}{3} \sqrt{y} dy = \frac{50\pi}{9} h^{3/2}$$

1 point is earned for the correct $\frac{dV}{dh}$

1 point is earned for the correct $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ by chain rule

1 point is earned for the correct answer $\frac{dV}{dt} = 8$

$\frac{dV}{dt} = \frac{25}{3}\pi\sqrt{h}\frac{dh}{dt}$
$\frac{dV}{dt} = 8$

when $h = 4, 8 = \frac{25}{3}\pi(2)\frac{dh}{dt}$

1 point is earned for the correct answer with units

Note: if *V* is linear, max 2/5 (0 - 0 - 1 - 1 - 0)

if *V* is constant, max 1/5 (0 - 0 - 0 - 1 - 0)

$\frac{dh}{dt} = \frac{12}{25\pi}$ ft/min

(or 0.152 ft/min or 0.153 ft/min)

# Week 6 Related Rates

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|

The student response earns five of the following points:

1 point is earned for the correct volume as definite integral using $h$

Finding $\frac{dV}{dt}$

$$V = \pi \int_0^h \frac{25}{3} \sqrt{y} dy = \frac{50\pi}{9} h^{3/2}$$

1 point is earned for the correct $\frac{dV}{dh}$

1 point is earned for the correct by $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ chain rule

1 point is earned for the correct answer $\frac{dV}{dt} = 8$

$\frac{dV}{dt} = \frac{25}{3}\pi\sqrt{h}\frac{dh}{dt}$
$\frac{dV}{dt} = 8$

when $h = 4, 8 = \frac{25}{3}\pi(2)\frac{dh}{dt}$

1 point is earned for the correct answer with units

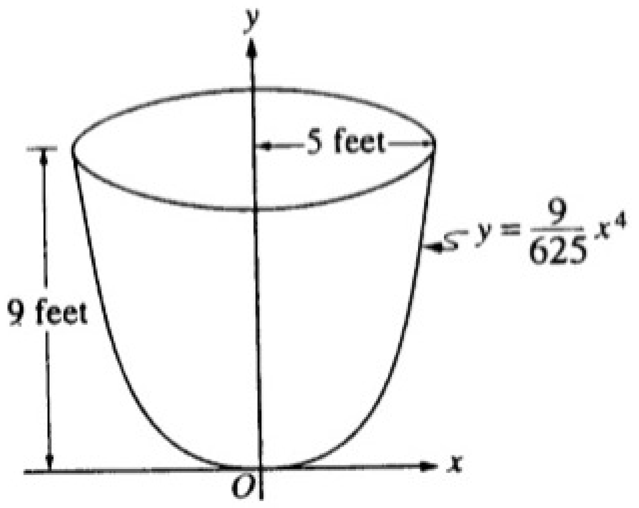
Note: if $V$ is linear, max 2/5 (0 - 0 - 1 - 1 - 0)

if $V$ is constant, max 1/5 (0 - 0 - 0 - 1 - 0)

$\frac{dh}{dt} = \frac{12}{25\pi}$ ft/min

(or 0.152 ft/min or 0.153 ft/min)

## Week 6 Related Rates




An oil storage tank has the shape as shown above, obtained by revolving the curve $y = \frac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the $y$-axis, where $x$ and $y$ are measured in feet.

Oil weighing 50 pounds per cubic foot flowed into an initially empty tank at a constant rate of 8 cubic feet per minute. When the depth of oil reached 6 feet, the flow stopped.

---

**17.** Let $h$ be the depth, in feet, of oil in the tank. How fast was the depth of oil in the tank increasing when $h = 4$? Indicate all units of measure.

Please respond on separate paper, following directions from your teacher.

**Part A**

1 point is earned for the correct volume as integral with $h$

$$V = \pi \int_0^h \frac{25}{3}\sqrt{y}dy$$

Finding $\frac{dV}{dt}$:

1 point is earned for the correct $\frac{dV}{dh}$

1 point is earned for the correct $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{25\pi}{3}\sqrt{h}\frac{dh}{dt}$

1 point is earned for the correct answer $\frac{dV}{dt} = 8$

## Week 6 Related Rates

at $h = 4$, $8 = \frac{25\pi}{3}\sqrt{4}\frac{dh}{dt}$

1 point is earned for the correct answer with units

$\frac{dh}{dt} = \frac{12}{25\pi}\text{ft}/\min$

Note: If $V$ is linear, max 2/5 (0 0 1 1 0)

If $V$ is constant, max 1/5 (0 0 0 1 0)

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|

The student response earns five of the following points:

1 point is earned for the correct volume as integral with $h$

$$V = \pi\int_0^h \frac{25}{3}\sqrt{y}dy$$

Finding $\frac{dV}{dt}$:

1 point is earned for the correct $\frac{dV}{dh}$

1 point is earned for the correct $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$

$\frac{dV}{dt} = \frac{25\pi}{3}\sqrt{h}\frac{dh}{dt}$

1 point is earned for the correct answer $\frac{dV}{dt} = 8$

at $h = 4$, $8 = \frac{25\pi}{3}\sqrt{4}\frac{dh}{dt}$

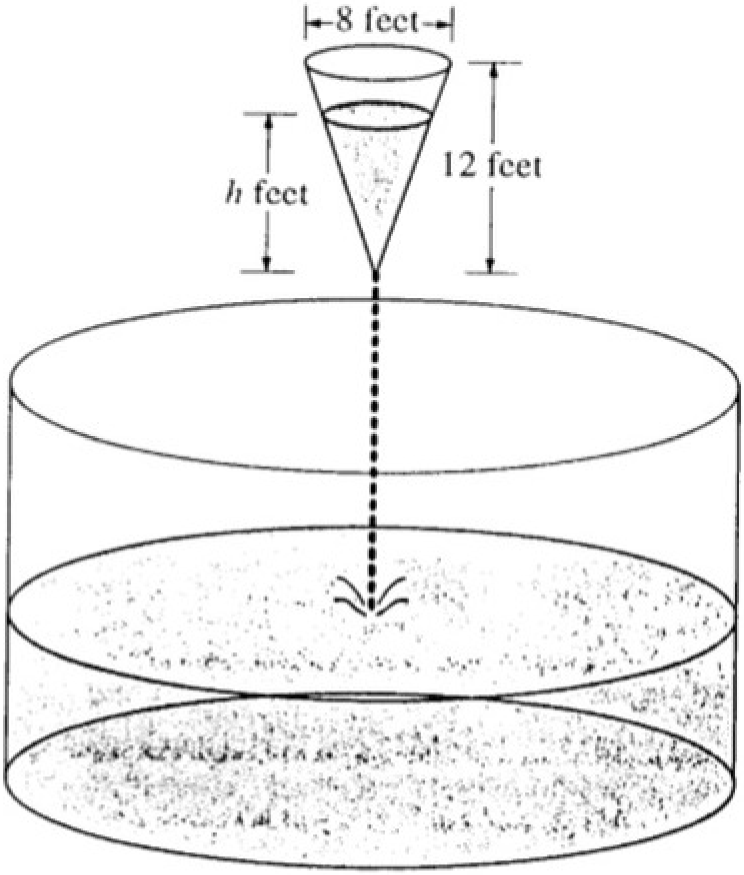
1 point is earned for the correct answer with units

$\frac{dh}{dt} = \frac{12}{25\pi}\text{ft}/\min$

Note: If $V$ is linear, max 2/5 (0 0 1 1 0)

If $V$ is constant, max 1/5 (0 0 0 1 0)

## Week 6 Related Rates




As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth $h$, in feet, of the water in the conical tank is changing at the rate of ($h$–12) feet per minute. (The volume $V$ of a cone with radius $r$ and height $h$ is $V = \frac{1}{3}\pi r^2 h$.)

**18.** At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part B**

1 point is earned for correctly solving $\frac{dV}{dt}$ using the chain rule

$$\frac{dV}{dt} = \frac{\pi h^2}{9}\frac{dh}{dt}$$

1 point is earned for the correct answer with $\frac{dh}{dt} = h - 12$

## Week 6 Related Rates

$\frac{\pi h^2}{9}(h-12)=-9\pi$

1 point is earned for correctly solving for $\frac{dV}{dt}$ and gives answer with unit

$V$ is decreasing at $9\pi$ ft$^3$/min

Note: 0/1 if $\frac{dV}{dt}>0$

| 0 | 1 | 2 | 3 |
|---|---|---|---|

The student response earns none of the following points:

1 point is earned for correctly solving $\frac{dV}{dt}$ using the chain rule

$\frac{dV}{dt}=\frac{\pi h^2}{9}\frac{dh}{dt}$

1 point is earned for the correct answer with $\frac{dh}{dt}=h-12$

$\frac{\pi h^2}{9}(h-12)=-9\pi$

1 point is earned for correctly solving for $\frac{dV}{dt}$ and gives answer with unit

$V$ is decreasing at $9\pi$ ft$^3$/min

Note: 0/1 if $\frac{dV}{dt}>0$

**19.** Let $y$ be the depth, in feet, of the water in the cylindrical tank. At what rate is $y$ changing when $h=3$? Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

**Part C**

1 point is earned for correctly defining $W$ as a function of $y$

Let $W$ = volume of water in cylindrical tank

## Week 6 Related Rates

$W = 400\pi y; \frac{dW}{dt} = 400\pi \frac{dy}{dt}$

$400\pi \frac{dy}{dt} = 9\pi$

1 point is earned for the correct answer for $\frac{dW}{dt} = 400\pi \frac{dy}{dt}$

1 point is earned for correctly identifying $\frac{dW}{dt} = \left|\frac{dV}{dt}\right|$ or $-\frac{dV}{dt}$

1 point is earned for correctly solving for $\frac{dy}{dt}$ and gives answer with units

$y$ is increasing at $\frac{9}{400}$ ft/min

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|

The student response earns four of the following points:

1 point is earned for correctly defining $W$ as a function of $y$

Let $W$ = volume of water in cylindrical tank

$W = 400\pi y; \frac{dW}{dt} = 400\pi \frac{dy}{dt}$

$400\pi \frac{dy}{dt} = 9\pi$

1 point is earned for the correct answer for $\frac{dW}{dt} = 400\pi \frac{dy}{dt}$

1 point is earned for correctly identifying $\frac{dW}{dt} = \left|\frac{dV}{dt}\right|$ or $-\frac{dV}{dt}$

1 point is earned for correctly solving for $\frac{dy}{dt}$ and gives answer with units

$y$ is increasing at $\frac{9}{400}$ ft/min

**20.** Write an expression for the volume of water in a conical tank as a function of $h$.

Please respond on separate paper, following directions from your teacher.

**Part A**

## Week 6 Related Rates

1 point is earned for the correct answer $r = \frac{1}{3}h$

$\frac{r}{h} = \frac{4}{12} = \frac{1}{3}$
$r = \frac{1}{3}h$

1 point is earned for correctly showing $V$ as a function of $h$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi h^3}{27}$

Note: 0/2 if $r$ constant

| | | |
|---|---|---|
| 0 | 1 | 2 |

The student response earns two of the following points:

1 point is earned for the correct answer $r = \frac{1}{3}h$

$\frac{r}{h} = \frac{4}{12} = \frac{1}{3}$
$r = \frac{1}{3}h$

1 point is earned for correctly showing $V$ as a function of $h$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi h^3}{27}$

Note: 0/2 if $r$ constant