Week 7 Linearization

1. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Consider the differential equation $\frac{dy}{dx} = 10 - 2y$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 0$. Use the tangent line to approximate $f(0.5)$.

Please respond on separate paper, following directions from your teacher.

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(0, 2)$. Is the graph of $y = f(x)$ concave up or concave down at the point $(0, 2)$? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 2$.

Please respond on separate paper, following directions from your teacher.

(d) For the particular solution $y = f(x)$ found in part (c), find $\lim_{x \to \infty} f(x)$.

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Please respond on separate paper, following directions from your teacher.

Part A

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2 ✓

The student response accurately includes both of the criteria below.

- ☐ tangent line equation
- ☐ approximation

Solution:

$\frac{dy}{dx}\Big|_{(x,\ y)=(0,\ 2)} = 10 - 2 \cdot 2 = 6$

An equation for the line tangent to the graph of $y = f\,(x)$ at $x = 0$ is $y = 6x + 2$.

$f\,(0.5) \approx 6 \cdot 0.5 + 2 = 5$

Part B

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2 ✓

The student response accurately includes both of the criteria below.

- ☐ $\frac{d^2y}{dx^2}\Big|_{(x,\ y)=(0,\ 2)}$
- ☐ concave down with reason

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Solution:

$$\frac{d^2y}{dx^2} = -2\frac{dy}{dx} = -2\left(10 - 2y\right) = 4y - 20$$

$$\left.\frac{d^2y}{dx^2}\right|_{(x,\ y)=(0,\ 2)} = 4 \cdot 2 - 20 = -12$$

Because $\left.\frac{d^2y}{dx^2}\right|_{(x,\ y)=(0,\ 2)} < 0$ and $\frac{d^2y}{dx^2}$ is continuous, the graph of $y = f\left(x\right)$ is concave down at the point $\left(0,\ 2\right)$.

Part C

Zero out of 4 points earned if no separation of variables.

At most 2 out of 4 points earned [1-1-0-0] if no constant of integration.

The fourth point requires an expression for y.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3	4 ✓

The student response accurately includes all four of the criteria below.

- ☐ separation of variables
- ☐ antiderivative
- ☐ constant of integration and uses initial condition
- ☐ solves for y

Solution:

$$\frac{1}{10-2y}\ dy = dx$$

$$\int \frac{1}{10-2y}\ dy = \int 1\ dx$$

$$-\frac{1}{2}\ln\left|10 - 2y\right| = x + C$$

Since the solution curve includes the point $\left(0,\ 2\right)$,

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$-\frac{1}{2}\ln(10-2y) = x + C$

$-\frac{1}{2}\ln(10-2\cdot 2) = 0 + C \Rightarrow C = -\frac{1}{2}\ln 6$

$-\frac{1}{2}\ln(10-2y) = x - \frac{1}{2}\ln 6$

$\ln(10-2y) = -2x + \ln 6$

$10 - 2y = e^{-2x+\ln 6} = 6e^{-2x}$

$y = 5 - 3e^{-2x}$

Part D

No supporting work is required to earn the point.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1

The student response accurately includes a correct value.

Solution:

$\lim_{x\to\infty}\left(5-3e^{-2x}\right) = 5 - 3\cdot 0 = 5$

Let f be the function defined by $f(x)=(1+\tan x)^{3/2}$ for $-\pi/4 < x < \pi/2$.

2. Using the equation found in part (a), approximate $f(0.02)$.

Please respond on separate paper, following directions from your teacher.

Part B

1 point is earned for evaluating their equation from part (a) at x=0.02

$f(0.02) \approx \frac{3}{2}(0.02) + 1$

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0	1 ✓

The student response earns one of the following points:

1 point is earned for evaluating their equation from part (a) at x=0.02

$f(0.02) \approx \frac{3}{2}(0.02) + 1$

The following are related to this scenario:

Let f be a function with $f(2) = -8$ such that for all points (x,y) on the graph of f, the slope given by $\frac{3x^2}{y}$.

3. Write an equation of the line tangent to the graph of f at the point where $x = 2$ and use it to approximate $f(1.8)$.

Please respond on separate paper, following directions from your teacher.

Part A

The response can earn up to 3 points:

1 point: For correct slope

Slope $\frac{(3)(4)}{-8} = -\frac{3}{2}$

1 point: For the tangent line equation

An equation for the tangent line is $y = -\frac{3}{2}(x-2) - 8$

1 point: For the correct approximation

$f(1.8) \approx -\frac{3}{2}(1.8-2) - 8 = -7.7$

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0	1	2	3

The response earns all three of the following points:

1 point: For correct slope

Slope $\frac{(3)(4)}{-8} = -\frac{3}{2}$

1 point: For the tangent line equation

An equation for the tangent line is $y = -\frac{3}{2}(x-2) - 8$

1 point: For the correct approximation

$f(1.8) \approx -\frac{3}{2}(1.8-2) - 8 = -7.7$

t (days)	0	10	22	30
$W'(t)$ (GL per day)	0.6	0.7	1.0	0.5

The twice-differentiable function W models the volume of water in a reservoir at time t, where $W(t)$ is measured in (GL) and t is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \le t \le 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

4. Use the tangent line approximation to W at time $t = 30$ to predict the volume of water W(t), in gigaliters, in the reservoir at time $t = 32$. Show the computations that lead to your answer.

Please respond on separate paper, following directions from your teacher.

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Part A

The response can earn up to 1 point:

1 point: For the correct answer

An equation of the tangent line is $y = 0.5(t - 30) + 125$.

$W(32) \approx 0.5(32\text{-}30) + 125 = 126$

0	1 ✓

The response earns the following point:

1 point: For the correct answer

An equation of the tangent line is $y = 0.5(t - 30) + 125$.

$W(32) \approx 0.5(32\text{-}30) + 125 = 126$

The function g is defined for $x>0$ with $g(1)=2$, $g'(x)= \sin(x+1/x)$, and $g''(x)=(1-1/x^2)\cos(x+1/x)$.

5. Does the line tangent to the graph of g at $x=0.3$ lie above or below the graph of g for $0.3<x<1$? Why?

Please respond on separate paper, following directions from your teacher.

Part D

1 point is earned for answer with reason

$g''(x)>0$ for $0.3<x<1$Therefore the line tangent to the graph of g at $x=0.3$ lies below the graph of g for $0.3<x<1$.

0	1 ✓

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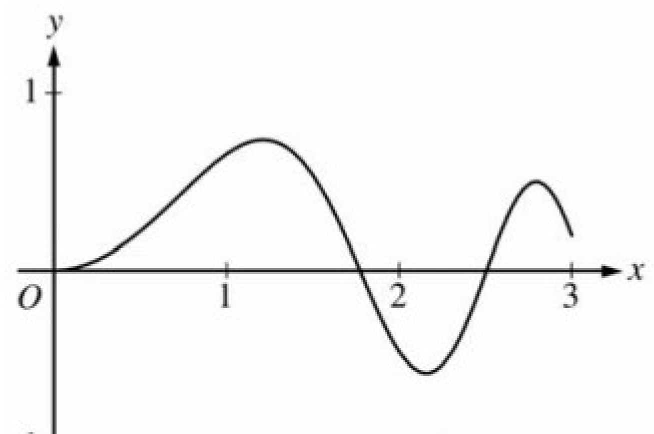
The student response earns one of the following points:

1 point is earned for answer with reason

$g''(x)>0$ for $0.3<x<1$

Therefore the line tangent to the graph of g at $x=0.3$ lies below the graph of g for $0.3<x<1$.



Graph of f'

Let f be the function defined for $x \geq 0$ with $f(0)=5$ and f', the first derivative of f, given by $f'(x)= e^{(-x/4)}\}\sin(x^2)$. The graph of $y = f'(x)$ is shown above.

6. Write an equation for the line tangent to the graph of f at $x=2$.

Please respond on separate paper, following directions from your teacher.

Part C

1 point is earned for $f(2)$ expression with integral

1 point is earned for $f(2)$ expression including f(0) term

1 point is earned for $f'(2)$

1 point is earned for equation

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$f(2)=f(0)+\int_0^2 f'(x)dx = 5.62342 f'(2)=e^{-0.5}\sin(4) = -0.45902 y-5.623 = (-0.459)(x-2)$

0	1	2	3	4 ✓

The student response earns all of the following points:

1 point is earned for $f(2)$ expression with integral

1 point is earned for $f(2)$ expression including f(0) term

1 point is earned for $f'(2)$

1 point is earned for equation

$f(2)=f(0)+\int_0^2 f'(x)dx = 5.62342$

$f'(2)=e^{-0.5}\sin(4) = -0.45902$

$y-5.623 = (-0.459)(x-2)$

Consider the curve defined by $-8x^2 + 5xy + y^3 = -149$.

7. Write an equation for the line tangent to the curve at the point $(4, -1)$.

Please respond on separate paper, following directions from your teacher.

Part B

1 point is earned for correctly evaluating $\frac{dy}{dx}$ at $(4, -1)$

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$\frac{dy}{dx}\Big|_{(4,-1)} = \frac{64+5}{20+3} = 3$

1 point is earned for the correct equation of tangent line

$y + 1 = 3\,(x - 4)$

✓

0	1	2

The student response earns two of the following points:

1 point is earned for correctly evaluating $\frac{dy}{dx}$ at $(4, -1)$

$\frac{dy}{dx}\Big|_{(4,-1)} = \frac{64+5}{20+3} = 3$

1 point is earned for the correct equation of tangent line

$y + 1 = 3\,(x - 4)$

8. There is a number k so that the point $(4.2, k)$ is on the curve. Using the tangent line found in part (b), approximate the value of k.

Please respond on separate paper, following directions from your teacher.

Part C

1 point is earned for correctly using $x = 4.2$ in tangent line equation

$y + 1 = 3\,(4.2 - 4)$

1 point is earned for the correct solution

< -1 > if "k ="

$$y = -0.4$$
$$k \approx -0.4$$

Week 7 Linearization

0	1	2 ✓

The student response earns two of the following points:

1 point is earned for correctly using $x = 4.2$ in tangent line equation

$y + 1 = 3\,(4.2 - 4)$

1 point is earned for the correct solution

< -1 > if "k ="

$y = -0.4$
$k \approx -0.4$

9. (d) Find $\lim_{x \to -2} \frac{f(x)+7}{e^{3x+6}-1}$.

Please respond on separate paper, following directions from your teacher.

(c) For each of $\lim_{x \to 0^-} g'(x)$ and $\lim_{x \to 0^+} g'(x)$, find the value or state that it does not exist.

Please respond on separate paper, following directions from your teacher.

(b) Find the value of x in the closed interval $[-4,\ 3]$ at which f attains its maximum value. Justify your answer.

Please respond on separate paper, following directions from your teacher.

CALCULUS AB

SECTION II, Part B

Time - 1 hour

Number of questions - 4

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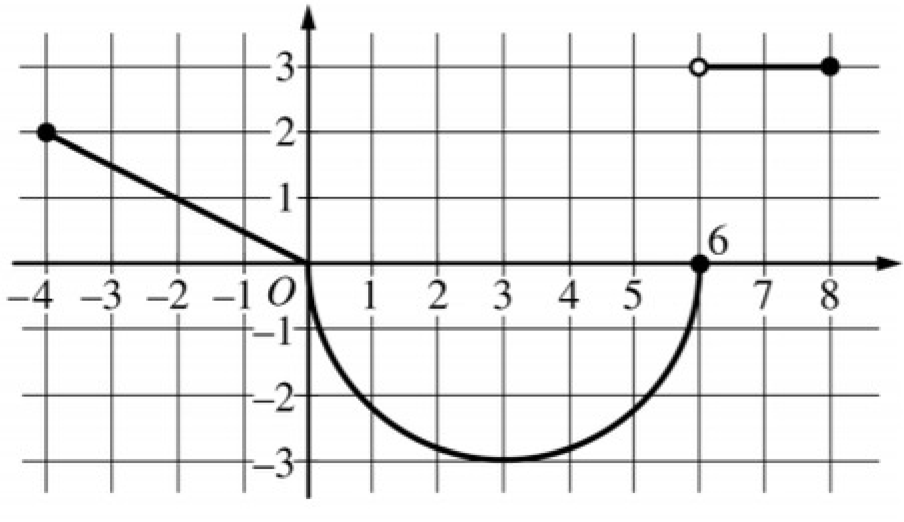
NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of g

3. The function g is defined on the closed interval $[-4,\ 8]$. The graph of g consists of two linear pieces and a semicircle, as shown in the figure above. Let f be the function defined by
$f(x) = 3x + \int_0^x g(t)dt.$

(a) Find $f\ (7)$ and $f'\ (7)$.

Please respond on separate paper, following directions from your teacher.

Part A

1 point is earned for: $f(7)$

1 point is earned for: $f'(7)$

$$f(7) = 3 \cdot 7 + \int_0^7 g(t)\ dt = 21 - \tfrac{9\pi}{2} + 3 = 24 - \tfrac{9\pi}{2}$$
$$f'(7) = 3 + g(7) = 3 + 3 = 6$$

0	1	2

The student response earns all of the following points:

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1 point is earned for: $f(7)$

1 point is earned for: $f'(7)$

$$f(7) = 3 \cdot 7 + \int_0^7 g(t)\,dt = 21 - \tfrac{9\pi}{2} + 3 = 24 - \tfrac{9\pi}{2}$$

$$f'(7) = 3 + g(7) = 3 + 3 = 6$$

Part B

2 points are earned for: answer with justification

On the interval $-4 \le x \le 3$, $f'(x) = 3 + g(x)$.

Because $f'(x) \ge 0$ for $-4 \le x \le 3$, f is nondecreasing over the entire interval, and the maximum must occur when $x = 3$.

0	1	2 ✓

The student response earns all of the following points:

2 points are earned for: answer with justification

On the interval $-4 \le x \le 3$, $f'(x) = 3 + g(x)$.

Because $f'(x) \ge 0$ for $-4 \le x \le 3$, f is nondecreasing over the entire interval, and the maximum must occur when $x = 3$.

Part C

1 point is earned for: left-hand limit

1 point is earned for: right-hand limit

$$\lim_{x \to 0^-} g'(x) = -\tfrac{1}{2}$$

$\lim_{x \to 0^+} g'(x)$ does not exist.

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0	1	2 ✓

The student response earns all of the following points:

1 point is earned for: left-hand limit

1 point is earned for: right-hand limit

$\lim_{x\to 0^-} g'(x) = -\frac{1}{2}$

$\lim_{x\to 0^+} g'(x)$ does not exist.

Part D

1 point is earned for: limits equal 0

1 point is earned for: applies L'Hospital's Rule

1 point is earned for: answer

$$\lim_{x\to -2}(f(x)+7) = -6 + \int_0^{-2} g(t)\,dt + 7 = 0$$
$$\lim_{x\to -2}(e^{3x+6}-1) = 0$$

Using L'Hospital's Rule,

$$\lim_{x\to -2}\frac{f(x)+7}{e^{3x+6}-1} = \lim_{x\to -2}\frac{f'(x)}{3e^{3x+6}} = \frac{3+g(-2)}{3} = \frac{3+1}{3} = \frac{4}{3}.$$

Note: max $1/3$ [1-0-0] if no limit notation attached to a ratio of derivatives

0	1	2	3 ✓

The student response earns all of the following points:

1 point is earned for: limits equal 0

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1 point is earned for: applies L'Hospital's Rule

1 point is earned for: answer

$$\lim_{x \to -2} (f(x) + 7) = -6 + \int_0^{-2} g(t)\, dt + 7 = 0$$
$$\lim_{x \to -2} (e^{3x+6} - 1) = 0$$

Using L'Hospital's Rule,

$$\lim_{x \to -2} \frac{f(x)+7}{e^{3x+6}-1} = \lim_{x \to -2} \frac{f'(x)}{3e^{3x+6}} = \frac{3+g(-2)}{3} = \frac{3+1}{3} = \frac{4}{3}.$$

Note: max $1/3$ [1-0-0] if no limit notation attached to a ratio of derivatives

10. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

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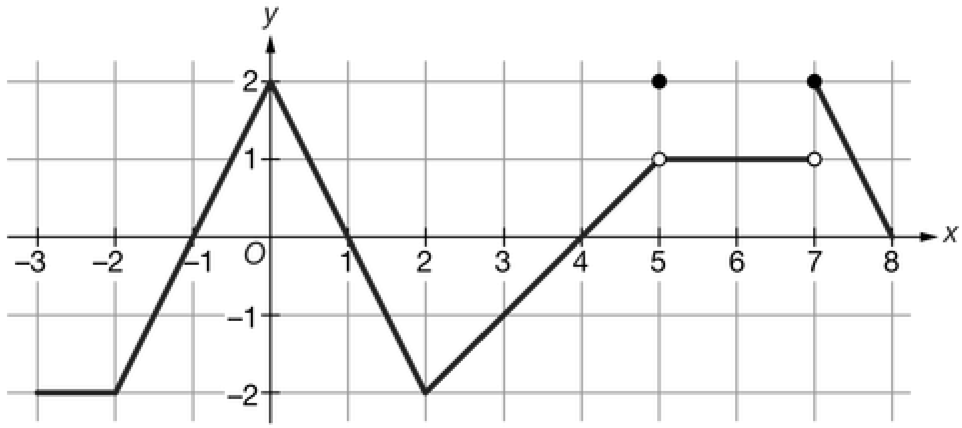

Graph of f

The graph of the function f on the closed interval $-3 \le x \le 8$ consists of six line segments and the point $(5, 2)$, as shown in the figure above. The function g is given by

$g(x) = \frac{1}{10}\left(4x^3 + 3x^2 - 10x - 17\right)$. It is known that $\int_{-3}^{-1} g(x)\,dx = -4.8$ and

$\int_{-3}^{4} g(x)\,dx = 11.2$.

(a) Find the value of $\int_{4}^{8} f(x)\,dx$, or explain why the integral does not exist.

Please respond on separate paper, following directions from your teacher.

(b)

(i) Find the value of $\int_{-1}^{4} g(x)\,dx$. Show the work that leads to your answer.

(ii) Find the value of $\int_{-1}^{4} \left(2g(x) - 4f(x)\right)\,dx$. Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

(c) Let $h(x) = \begin{cases} g(x) & \text{for } x \le -1 \\ f(x) + b & \text{for } x > -1 \end{cases}$. Find the value of b for which $\int_{-3}^{4} h(x)\,dx = 14.2$.

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Please respond on separate paper, following directions from your teacher.

(d) Find $\lim_{x \to 1} \frac{f(x)}{g(x)+2}$. Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

Part A

The response should include supporting work for the numerical answer of 3.5.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1 ✓

The student response accurately includes the value of the integral.

Solution:

The integral exists since f is continuous on the interval $[4,\ 8]$, except for a removable discontinuity at $x = 5$ and a jump discontinuity at $x = 7$.

Using areas, $\int_4^8 f\ (x)\ \ dx = 0.5 + 2 + 1 = 3.5.$

Part B

Answers without supporting work do not earn any points.

A response that includes a sign error in an attempt to find $\int_{-1}^{4} g\ (x)\ \ dx$ is eligible for the second and third points.

A response that includes a sign error in an attempt to find $\int_{-1}^{4} f\ (x)\ \ dx$ is eligible for the second point and not eligible for the third point.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

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0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ $\int_{-1}^{4} g(x)\ dx$
- ☐ uses integral properties
- ☐ answer

Solution:

(i) $\int_{-1}^{4} g(x)\ dx = \int_{-3}^{4} g(x)\ dx - \int_{-3}^{-1} g(x)\ dx = 11.2 - (-4.8) = 16$

(ii) Using areas, $\int_{-1}^{4} f(x)\ dx = 2 - 3 = -1.$

$$\int_{-1}^{4} (2g(x) - 4f(x))\ dx = 2\int_{-1}^{4} g(x)\ dx - 4\int_{-1}^{4} f(x)\ dx$$
$$= 2(16) - 4(-1) = 32 + 4 = 36$$

Part C

The evaluation of the definite integrals is assessed in the second point.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2 ✓

The student response accurately includes both of the criteria below.

- ☐ uses integral properties
- ☐ answer

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Solution:

Using properties of integrals,

$$\int_{-3}^{4} h(x)\ dx = \int_{-3}^{-1} g(x)\ dx + \int_{-1}^{4} (f(x) + b)\ dx$$
$$= \int_{-3}^{-1} g(x)\ dx + \int_{-1}^{4} f(x)\ dx + \int_{-1}^{4} b\ dx$$
$$= -4.8 + (-1) + (4 - (-1))\, b = -5.8 + 5b$$

$-5.8 + 5b = 14.2 \ \Rightarrow\ 5b = 20\ \Rightarrow b = 4$

Part D

For the first point, a response must show that $\lim\limits_{x\to1} f(x) = 0$ and $\lim\limits_{x\to1} (g(x) + 2) = 0$ and present an attempt at a limit of a ratio of derivatives. Use of $\frac{0}{0}$ in an equality statement with an expression or value is incorrect and results in the point not being earned.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ L'Hospital's Rule
- ☐ $g'(x)$
- ☐ answer

Solution:

$\lim\limits_{x\to1} f(x) = 0$ and $\lim\limits_{x\to1} (g(x) + 2) = -2 + 2 = 0.$

Using L'Hospital's Rule,

$$\lim_{x\to1} \frac{f(x)}{g(x)+2} = \lim_{x\to1} \frac{f'(x)}{g'(x)} = \lim_{x\to1} \frac{f'(x)}{\frac{1}{10}(12x^2+6x-10)}$$
$$= \frac{-2}{\frac{1}{10}(12+6-10)} = \frac{-20}{8} = -\frac{5}{2}.$$

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The function g is continuous for all real numbers x and is defined by

$$g(x) = \frac{\cos(2x)-1}{x^2} \; for \; x \neq 0.$$

11. Use L'Hospital's Rule to find the value of $g(0)$. Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

Part A

One point is earned for uses L'Hospital's rule correctly at least once

One point is earned for the answer

$$g(0) = \lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\cos(2x)-1}{x^2}$$
$$= \lim_{x\to 0} \frac{-2\sin(2x)}{2x}$$
$$= \lim_{x\to 0} \frac{-4\cos(2x)}{2} = -2$$

0	1	2

The student earns all of the following points:

One point is earned for uses L'Hospital's rule correctly at least once

One point is earned for the answer

$$g(0) = \lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\cos(2x)-1}{x^2}$$
$$= \lim_{x\to 0} \frac{-2\sin(2x)}{2x}$$
$$= \lim_{x\to 0} \frac{-4\cos(2x)}{2} = -2$$

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12. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Particle P moves along the y-axis so that its position at time t is given by $y(t) = 4t - \frac{2}{3}$ for all times t. A second particle, particle Q, moves along the x-axis so that its position at time t is given by $x(t) = \frac{\sin(\pi t)}{2-t}$ for all times $t \neq 2$.

(a) As time t approaches 2, what is the limit of the position of particle Q ? Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

(b) Show that the velocity of particle Q is given by $v_Q(t) = \frac{2\pi \cos(\pi t) - \pi t \cos(\pi t) + \sin(\pi t)}{(2-t)^2}$ for all times $t \neq 2$.

Please respond on separate paper, following directions from your teacher.

(c) Find the rate of change of the distance between particle P and particle Q at time $t = \frac{1}{2}$. Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

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Part A

The first point requires showing that an indeterminate form $\frac{0}{0}$ is present in order to apply L'Hospital's Rule. Any error in mathematical communication (e.g., writing "$= \frac{0}{0}$" or not using limit notation) impacts the second point.

The second point does not require a simplified answer. Trigonometric function values do not need to be evaluated. Any differentiation or computation error impacts this point.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2 ✓

The student response accurately includes both of the criteria below.

- ☐ application of L'Hospital's Rule
- ☐ answer

Solution:

$$\lim_{t \to 2} \sin(\pi t) = 0$$

$$\lim_{t \to 2} (2 - t) = 0$$

By L'Hospital's Rule,

$$\lim_{t \to 2} \frac{\sin(\pi t)}{2-t} = \lim_{t \to 2} \frac{\pi \cos(\pi t)}{-1} = -\pi$$

Part B

The first and second points require evidence of quotient rule and chain rule and no errors. At most 1 out of 2 points is earned for partial communication of quotient rule and chain rule with a maximum of one computational error.

The third point is earned for a response that arrives at the given expression rather than an algebraically equivalent expression.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

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0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ quotient rule
- ☐ chain rule
- ☐ verification

Solution:

$$v_Q\left(t\right)=x'\left(t\right)=\frac{(2-t)\pi\cos(\pi t)-(-1)\sin(\pi t)}{(2-t)^2}=\frac{2\pi\cos(\pi t)-\pi t\cos(\pi t)+\sin(\pi t)}{(2-t)^2}$$

Part C

For the second point, trigonometric function values do not need to be evaluated.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3	4 ✓

The student response accurately includes all four of the criteria below.

- ☐ $y'\left(\frac{1}{2}\right)$
- ☐ $x'\left(\frac{1}{2}\right)$
- ☐ related rates equation $2D\frac{dD}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$
- ☐ answer

At time $t=\frac{1}{2}$, particle P has position $y\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)-\frac{2}{3}=2-\frac{2}{3}=\frac{4}{3}$ and velocity $y'\left(\frac{1}{2}\right)=v_P\left(\frac{1}{2}\right)=4$.

At time $t=\frac{1}{2}$, particle Q has position $x\left(\frac{1}{2}\right)=\frac{\sin\left(\frac{\pi}{2}\right)}{2-\frac{1}{2}}=\frac{1}{\frac{3}{2}}=\frac{2}{3}$ and velocity

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$$x'\left(\frac{1}{2}\right) = v_Q\left(\frac{1}{2}\right) = \frac{2\pi\cos\left(\frac{\pi}{2}\right)-\pi\left(\frac{1}{2}\right)\cos\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)}{\left(2-\frac{1}{2}\right)^2} = \frac{1}{\left(\frac{3}{2}\right)^2} = \frac{4}{9}.$$

Let D represent the distance between particle P and particle Q Using the Pythagorean Theorem, $D^2 = x^2 + y^2$.

Therefore, $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$.

$$\frac{dD}{dt} = \frac{2x\frac{dx}{dt}+2y\frac{dy}{dt}}{2D} = \frac{x\frac{dx}{dt}+y\frac{dy}{dt}}{D}$$

At time $t = \frac{1}{2}$,

$$D^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 \Rightarrow D^2 = \frac{20}{9} \Rightarrow D = \frac{\sqrt{20}}{3}.$$

$$\left.\frac{dD}{dt}\right|_{t=\frac{1}{2}} = \frac{\left(\frac{2}{3}\right)\left(\frac{4}{9}\right)+\left(\frac{4}{3}\right)(4)}{\frac{\sqrt{20}}{3}} = \frac{\frac{8}{27}+\frac{16}{3}}{\frac{\sqrt{20}}{3}} = \frac{152}{27}\cdot\frac{3}{\sqrt{20}} = \frac{152}{9\sqrt{20}}$$

The distance between the particles is increasing at a rate of $\frac{152}{9\sqrt{20}}$.

13. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

A particle moves along the y-axis so that its position at time t is given by $y(t) = t^2\tan\left(\frac{1}{t}\right)$ for $t > 1$.

(a) Show that the velocity of the particle at time t is given by $v(t) = 2t\tan\left(\frac{1}{t}\right) - \sec^2\left(\frac{1}{t}\right)$ for $t > 1$.

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Please respond on separate paper, following directions from your teacher.

(b) At time $t = \frac{4}{\pi}$, is the particle moving toward the origin or away from the origin? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

(c) The velocity of the particle at time t can be written as $v(t) = \frac{2\tan\left(\frac{1}{t}\right)}{\frac{1}{t}} - \sec^2\left(\frac{1}{t}\right)$ for $t > 1$.

Find $\lim_{t \to \infty} v(t)$. Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

Part A

The first and second points require evidence of product rule and chain rule and no errors. At most 1 out of 2 points is earned for partial communication of product rule and chain rule with a maximum of one computational error.

The third point is earned for a response that arrives at the given expression rather than an algebraically equivalent expression.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3

The student response accurately includes all three of the criteria below.

- ☐ product rule
- ☐ chain rule
- ☐ verification

Solution:

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$v(t) = y'(t) = 2t\tan\left(\frac{1}{t}\right) + t^2\sec^2\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right) = 2t\tan\left(\frac{1}{t}\right) - \sec^2\left(\frac{1}{t}\right)$

Part B

For the first and second points, trigonometric function values do not need to be evaluated; however, the sign of the expression must be determined correctly to earn the point.

The response is eligible for the third point if either of the first 2 points is earned and the answer with reason is consistent with the previous results.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ $y\left(\frac{4}{\pi}\right) > 0$
- ☐ $v\left(\frac{4}{\pi}\right) > 0$
- ☐ answer with reason

Solution:

$y\left(\frac{4}{\pi}\right) = \left(\frac{4}{\pi}\right)^2 \tan\left(\frac{\pi}{4}\right) = \frac{16}{\pi^2} > 0$

$v\left(\frac{4}{\pi}\right) = 2\left(\frac{4}{\pi}\right)\tan\left(\frac{\pi}{4}\right) - \sec^2\left(\frac{\pi}{4}\right) = \frac{8}{\pi} - 2 > 0$

Because $y\left(\frac{4}{\pi}\right) > 0$, the particle is above the origin at time $t = \frac{4}{\pi}$.

Because $v\left(\frac{4}{\pi}\right) > 0$, the particle is moving up at time $t = \frac{4}{\pi}$.

Therefore, the particle is moving away from the origin at time $t = \frac{4}{\pi}$.

Part C

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The first point does not require supporting work.

The second point requires showing that an indeterminate form $\frac{0}{0}$ is present in order to apply L'Hospital's Rule. Any error in mathematical communication (e.g., writing $= \frac{0}{0}$ or not using limit notation) impacts the second point.

The third point does not require a simplified answer. Any differentiation or computation error impacts the third point.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ $\lim_{t\to\infty} \sec^2\left(\frac{1}{t}\right) = 1$
- ☐ application of L'Hospital's Rule
- ☐ answer

Solution:

$$\lim_{t\to\infty} 2\tan\left(\frac{1}{t}\right) = 2\tan 0 = 0$$

$$\lim_{t\to\infty} \frac{1}{t} = 0$$

$$\lim_{t\to\infty} \sec^2\left(\frac{1}{t}\right) = \sec^2 0 = 1$$

By L'Hospital's Rule,

$$\lim_{t\to\infty} \frac{2\tan\left(\frac{1}{t}\right)}{\frac{1}{t}} = \lim_{t\to\infty} \frac{2\sec^2\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}{-\frac{1}{t^2}} = \lim_{t\to\infty} 2\sec^2\left(\frac{1}{t}\right) = 2$$

Therefore,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(\frac{2\tan\left(\frac{1}{t}\right)}{\frac{1}{t}} - \sec^2\left(\frac{1}{t}\right)\right) = 2 - 1 = 1$$

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14. Let f and g be functions that are differentiable for all real numbers, with $g(x) \neq 0$ for $x \neq 0$. If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to 0} \frac{f(x)}{g(x)}$ is

(A) 0

(B) $\frac{f'(x)}{g(x)}$

(C) $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ ✓

(D) $\frac{f'(x)g(x)-f(x)g'(x)}{(f(x))^2}$

(E) nonexistent

15. $\lim_{t\to 0} \frac{\sin t}{\ln(2e^t-1)} =$

(A) -1

(B) 0

(C) $\frac{1}{2}$ ✓

(D) 1

16. Let g be a continuously differentiable function with $g(1) = 6$ and $g'(1) = 3$. What is $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6}$?

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(A) 0

(B) $\frac{1}{2}$

(C) 1

(D) 2 ✓

(E) The limit does not exist.