Chapter 3 Polynomial Functions

3.1 Characteristics of Polynomial Functions

KEY IDEAS

What Is a Polynomial Function?

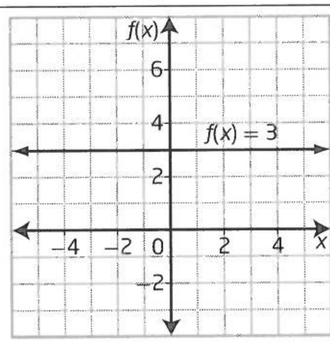
A polynomial function has the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$ where

- n is a whole number
- x is a variable
- the coefficients a_n to a_0 are real numbers
- the degree of the polynomial function is n, the exponent of the greatest power of x
- the leading coefficient is a_n, the coefficient of the greatest power of x
- the constant term is a_0

Types of Polynomial Functions

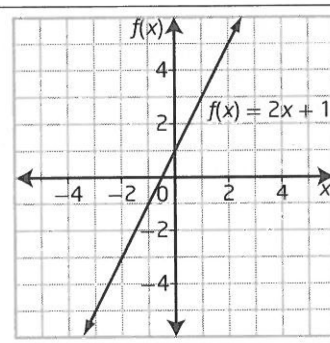
Constant Function

Degree 0



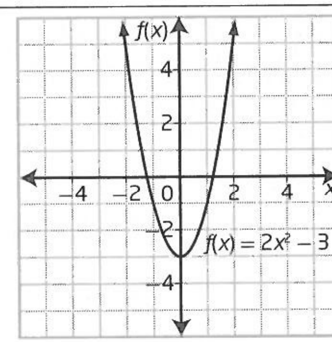
Linear Function

Degree 1



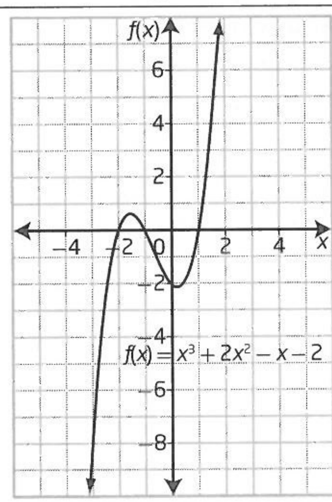
Quadratic Function

Degree 2



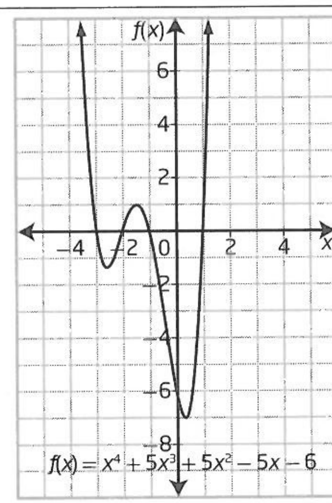
Cubic Function

Degree 3



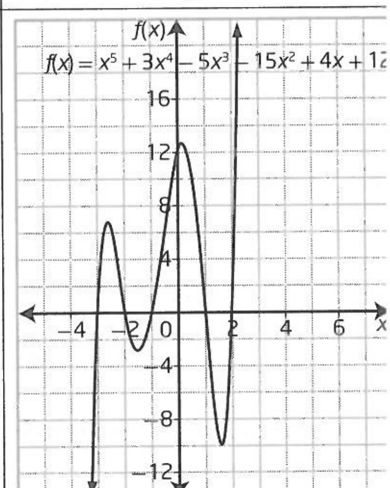
Quartic Function

Degree 4



Quintic Function

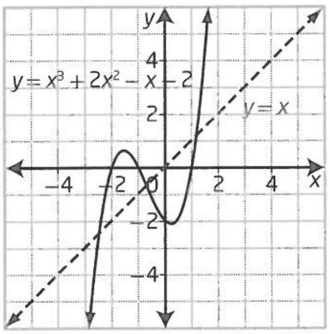
Degree 5



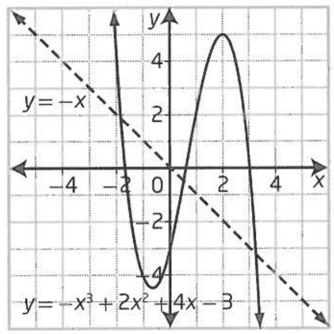
Characteristics of Polynomial Functions

Graphs of Odd-Degree Polynomial Functions

- extend from quadrant III to quadrant I when the leading coefficient is positive, similar to the graph of $y = x$



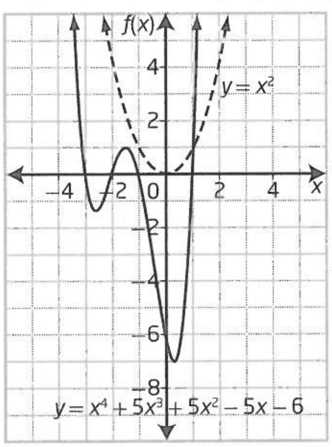
- extend from quadrant II to IV when the leading coefficient is negative, similar to the graph of $y = -x$



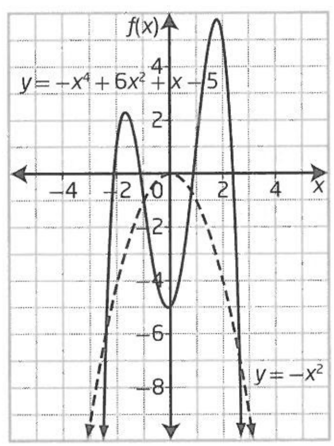
- have at least one x-intercept to a maximum of n x-intercepts, where n is the degree of the function
- have y-intercept a_0, the constant term of the function
- have domain $\{x \mid x \in \mathrm{R}\}$ and range $\{y \mid y \in \mathrm{R}\}$
- have no maximum or minimum values

Graphs of Even-Degree Polynomial Functions

- open upward and extend from quadrant II to quadrant I when the leading coefficient is positive, similar to the graph of $y = x^2$



- open downward and extend from quadrant III to IV when the leading coefficient is negative, similar to the graph of $y = -x^2$



- have from 0 to a maximum of n x-intercepts, where n is the degree of the function
- have y-intercept a_0, the constant term of the function
- have domain $\{x \mid x \in \mathrm{R}\}$; the range depends on the maximum or minimum value of the function
- have a maximum or minimum value

Working Example 1: Identify Polynomial Functions

Which of these functions are polynomials? Justify your answer. State the degree, the leading coefficient, and the constant term of each polynomial function.

a) $g(x) = 4^x$

b) $h(x) = 6 - 5x^3$

c) $y = x^{-2} + 1$

d) $y = \sqrt[3]{x} - 2$

Solution

a) $g(x) = 4^x$

The function $g(x) = 4^x$ ________ a polynomial function.
(*is* or *is not*)

It is an ________________ function.

The base of 4^x is a ________________ and the exponent is a ________________, so it is not of the form x^n.

b) $h(x) = 6 - 5x^3$

The function $h(x) = 6 - 5x^3$ ________ a polynomial function.
(*is* or *is not*)

The leading coefficient is ________, the degree is ________, and the constant term is ________.

c) $y = x^{-2} + 1$

The function $y = x^{-2} + 1$ ________ a polynomial function.
(*is* or *is not*)

x^{-2} is the same as ________, because the exponent is negative.

The function $y = x^{-2} + 1$ is a ________________ function.

d) $y = \sqrt[3]{x} - 2$

The function $y = \sqrt[3]{x} - 2$ ________ a polynomial function.
(*is* or *is not*)

$\sqrt[3]{x}$ can be rewritten as ________, which has a ________________ exponent.

The function $y = \sqrt[3]{x} - 2$ is a ________________ function.

Working Example 2: Match a Polynomial Function With Its Graph

For each polynomial function, identify the following characteristics:

- the type of function and whether it is of even or odd degree
- the end behaviour of the graph of the function
- the number of possible x-intercepts
- whether the graph has a maximum or minimum value
- the y-intercept

Then, match each function to its corresponding graph.

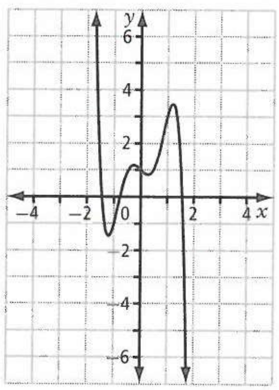
a) $f(x) = 2x^3 - 4x^2 + x + 2$

b) $g(x) = -x^4 + 10x^2 + 5x - 6$

c) $h(x) = -2x^5 + 5x^3 - x + 1$

d) $p(x) = x^4 - 5x^3 + 16$

A



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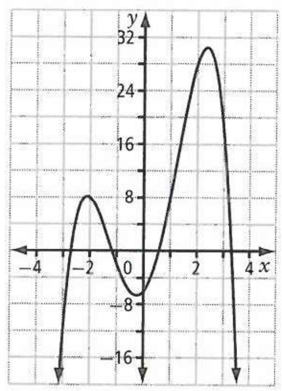
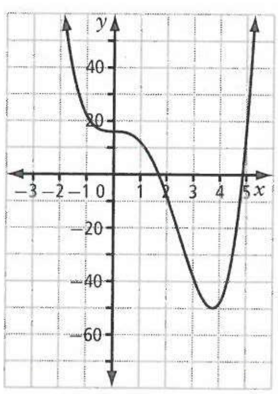


C



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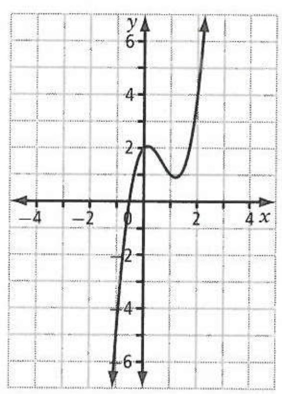


Solution

a) $f(x) = 2x^3 - 4x^2 + x + 2$ has degree ________, so it is an ________ function.
(*even* or *odd*)

This is a ________________ polynomial function.
(*constant* or *linear* or *quadratic* or *cubic* or *quartic* or *quintic*)

Its graph has at least ________ x-intercept(s) and at most ________ x-intercepts.

The leading coefficient is ________________, so the graph of the function extends from
(*positive* or *negative*)

quadrant ________ into quadrant ________.

Since the degree is ________, this function ______________ a maximum or minimum value.
(*even* or *odd*) (*has* or *does not have*)

The graph has a y-intercept of ________.

This function corresponds to graph ________.
(A or B or C or D)

b) $g(x) = -x^4 + 10x^2 + 5x - 6$ has degree ________, so it is an ________ function.
(*even* or *odd*)

This is a ______________ polynomial function.

Its graph has at least ________ x-intercept(s) and at most ________ x-intercepts.

Since the leading coefficient is ____________, the graph of the function opens ____________
(*positive* or *negative*) (*upward* or *downward*)

and has a ______________ value.
(*maximum* or *minimum*)

The graph has a y-intercept of ________.

This function corresponds to graph ________.

c) $h(x) = -2x^5 + 5x^3 - x + 1$ has degree ________, so it is an ________ function.
(*even* or *odd*)

This is a ______________ polynomial function.

Its graph has at least ________ x-intercept(s) and at most ________ x-intercepts.

The leading coefficient is ______________, so the graph of the function extends from
(*positive* or *negative*)

quadrant ________ into quadrant ________.

Since the degree is ________, this function ______________ a maximum or minimum value.
(*even* or *odd*) (*has* or *does not have*)

The graph has a y-intercept of ________.

This function corresponds to graph ________.

d) $p(x) = x^4 - 5x^3 + 16$ has degree ________, so it is an ________ function.
(*even* or *odd*)

This is a ______________ polynomial function.

Its graph has at least ________ x-intercept(s) and at most ________ x-intercepts.

The leading coefficient is ______________, so the graph of the function opens ____________
(*positive* or *negative*) (*upward* or *downward*)

and has a ______________ value.
(*maximum* or *minimum*)

The graph has a y-intercept of ________.

This function corresponds to graph ________.

Working Example 3: Application of a Polynomial Function

An antibacterial spray is tested on a bacterial culture. The population, P, of bacteria t minutes after the spray is applied is modelled by the function $P(t) = -2t^3 - 2t^2 + 3t + 800$.

a) What is the population of the bacteria 3 min after the spray is applied?

b) How many bacteria were in the culture before the spray was applied?

c) What is the population of the bacteria 8 min after the spray is applied? Why is this not realistic for this situation? Explain.

Solution

a) Substitute $t =$ ______ into the function and evaluate the result.

$P(\_\_\_\_\_\_) = -2(\_\_\_\_\_\_)^3 - 2(\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) + 800$

$= \_\_\_\_\_\_$

After 3 min there are ______ bacteria in the culture.

See page 112 of *Pre-Calculus 12* for a different method of solving this question.

b) The number of bacteria before the spray was applied occurs when $t =$ ______.

This is the ______-intercept of the graph.

It is the ______________ term of the function.
(*constant* or *variable*)

There were ______ bacteria before the spray was applied.

c) Substitute $t =$ ______ into the function and evaluate the result.

$P(\_\_\_\_\_\_) = -2(\_\_\_\_\_\_)^3 - 2(\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) + 800$

$= \_\_\_\_\_\_$

After 8 min there are ______ bacteria in the culture.

It is not realistic for the number of bacteria to be ______________.
(*positive* or *negative*)

This means that the spray has worked and there are ______ bacteria remaining.

Check Your Understanding

Practise

1. Determine whether each function is a polynomial function. Justify your answers.

   a) $f(x) = 2x^4 - 3x + 2$

   The degree is ________, which is an ________________ number.

   $f(x)$ ________ a polynomial function.
   (*is* or *is not*)

   b) $y = 3^x + 5$

   The term 3^x means this is an ________________ function.

   This function ________ a polynomial function.
   (*is* or *is not*)

   c) $g(x) = 9$

   $g(x)$ has degree ________.

   This function ________ a polynomial function.
   (*is* or *is not*)

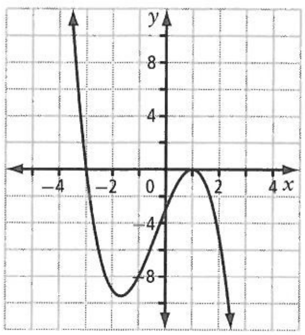
   d) $y = x^{-2} + 7x^3 + 1$

2. Complete the table for each polynomial function.

Polynomial Function	Degree	Type	Leading Coefficient	Constant Term
a) $f(x) = 6x^3 - 5x^2 + 2x - 8$	3			
b) $y = -2x^5 + 5x^3 + x^2 + 1$		Quintic		
c) $g(x) = x^3 - 7x^4$				0
d) $p(x) = 10x - 9$				
e) $y = -0.5x^2 + 4x + 3$				
f) $h(x) = 3x^4 - 8x^3 + x^2 + 2$			3	
g) $y = -5$		Constant		

3. For each graph of a polynomial function,
   - determine whether the function has odd or even degree
   - determine whether the leading coefficient is positive or negative
   - state the number of x-intercepts
   - state the domain and range

a) $f(x) = -x^3 - x^2 + 5x - 3$



The graph extends from quadrant ______ to quadrant ______.

The function has ______ degree.

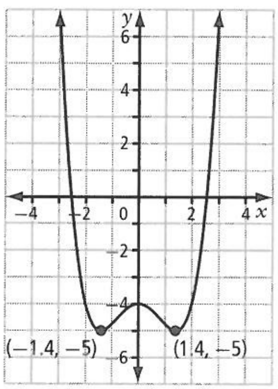
The leading coefficient is ______.

There are ______ x-intercepts.

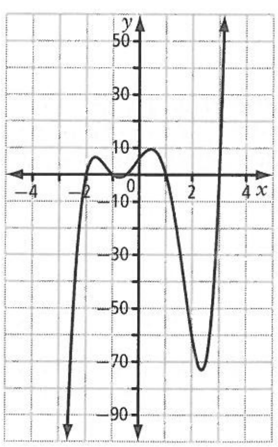
Domain: ______________

Range: ______________

b)



c)



4. For each function, use the degree and the sign of the leading coefficient to describe the end behaviour of its graph. State the possible number of x-intercepts and the value of the y-intercept.

   a) $g(x) = 4x^5 - x^3 + 3x^2 - 6x + 2$

      The degree is ________ with a ________________ leading coefficient.

      The graph extends from quadrant ________ to quadrant ________.

      There are a maximum of ________ x-intercepts. The y-intercept is ________.

   b) $y = -x^4 - 2x^5 + x^3 - 3x^2 + x$

   c) $h(x) = x - 7x^3 - 6$

   d) $y = 3x^5 - 2x^4 + 5x^3 - x^2 + x + 3$

   e) $p(x) = 5x^4 - 6x - 1$

Apply

5. Sonja claims that all graphs of polynomial functions of the form $y = ax^n + x + b$, where a, n, and b are odd integers, extend from quadrant II to quadrant IV. Do you agree? Use examples to explain your answer.

5. A skateboard manufacturer determines that its profit, P, in dollars, can be modelled by the function $P(x) = 1000x + 1.25x^4 - 3200$, where x represents the number, in hundreds, of skateboards sold.

a) What is the degree of the function $P(x)$?

b) What are the leading coefficient and the constant of this function? What does the constant represent in this context?

c) Describe the end behaviour of the graph of this function.

d) What are the restrictions on the domain of this function? Explain how you determined those restrictions.

e) What do the x-intercept(s) of the graph represent in this context?

f) What is the profit from the sale of 1200 skateboards?

7. Ali moves forward and backward along a straight path. Ali's distance, D, in metres, from a tree is modelled by the function $D(t) = t^3 - 12t^2 + 36t + 5$, where t represents the time, in seconds.

a) What is the degree of function $D(t)$?

b) What are the leading coefficient and the constant of this function? What does the constant represent in this situation?

c) Describe the end behaviour of the graph of this function.

d) What are the restrictions on the domain of this function? Explain how you determined the restrictions.

e) How far is Ali from the tree after 7 s?

f) Sketch the function. Then, graph the function using technology. How does the graph compare to your sketch?

8. By analysing the effect of growing economic conditions, the predicted population, P, of a town in t years from now can be modelled by the function $P(t) = 6t^4 - 6t^3 + 200t + 12\,000$. Assume this model can be used for the next 15 years.

a) What are the key features of the graph of this function?

b) What is the current population of this town?

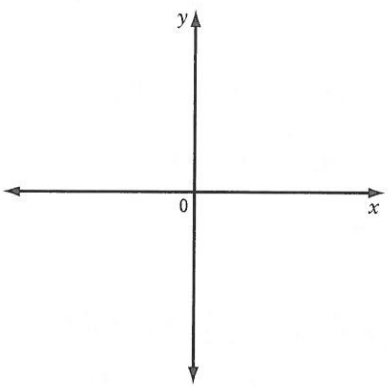
c) What will the population be 10 years from now?

d) When will the population of the town be approximately 175 000?

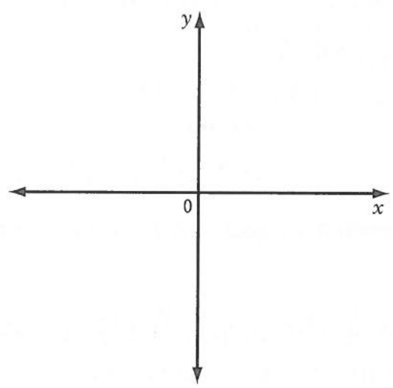
Connect

On each set of axes, sketch a polynomial function with the given characteristics.

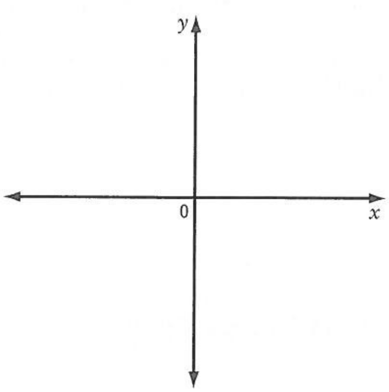
a) A polynomial function with degree 3, a positive leading coefficient, and 2 x-intercepts.



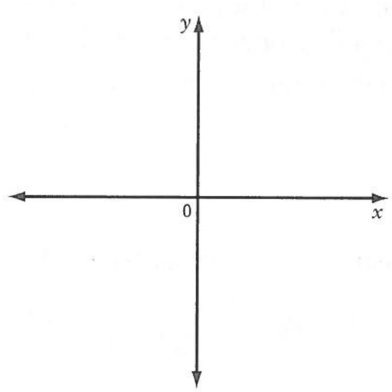
b) A polynomial function with degree 4, a negative leading coefficient, and 4 x-intercepts.



c) A polynomial function with degree 5, a negative leading coefficient, and 3 x-intercepts.



d) A polynomial function with degree 4, a positive leading coefficient, and 2 x-intercepts.



Chapter 3

3.1 Characteristics of Polynomial Functions, pages 66–77

1. a) Yes; polynomial function of degree 4
   b) No; exponential function
   c) Yes; polynomial function of degree 0
   d) No; function has a variable with a negative exponent

2.

Polynomial Function	Degree	Type	Leading Coefficient	Constant Term
a) $f(x) = 6x^3 - 5x^2 + 2x - 8$	3	Cubic	6	−8
b) $y = -2x^5 + 5x^3 + x^2 + 1$	5	Quintic	−2	1
c) $g(x) = x^3 - 7x^4$	4	Quartic	−7	0
d) $p(x) = 10x - 9$	1	Linear	10	−9
e) $y = -0.5x^2 + 4x + 3$	2	Quadratic	−0.5	3
f) $h(x) = 3x^4 - 8x^3 + x^2 + 2$	4	Quartic	3	2
g) $y = -5$	0	Constant	0	−5

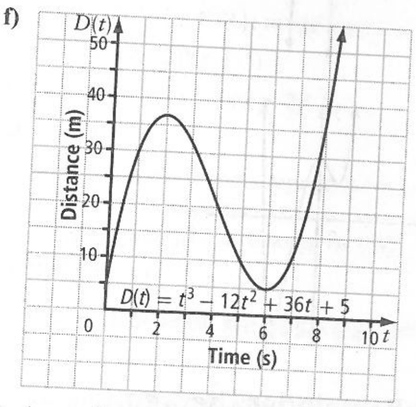
3. a) odd degree; negative leading coefficient; 2 x-intercepts; domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \in \mathrm{R}\}$
   b) even degree; positive leading coefficient; 2 x-intercepts; domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \geq -5, y \in \mathrm{R}\}$
   c) odd degree; positive leading coefficient; 5 x-intercepts; domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \in \mathrm{R}\}$

4. a) degree 5; positive leading coefficient; extends from quadrant III to I; maximum of 5 x-intercepts; y-intercept of 2
   b) degree 5; negative leading coefficient; extends from quadrant II to IV; maximum of 5 x-intercepts; y-intercept of 0
   c) degree 3; negative leading coefficient; extends from quadrant II to IV; maximum of 3 x-intercepts; y-intercept of −6
   d) degree 5; positive leading coefficient; extends from quadrant III to I; maximum of 5 x-intercepts; y-intercept of 3
   e) degree 4; positive leading coefficient; opens upward; maximum of 4 x-intercepts; y-intercept of −1

5. No. Example: $y = 5x^3 + x + 1$ extends from quadrant III to I

6. a) degree 4
   b) leading coefficient: 1.25; constant: −3200; The constant represents the initial cost.
   c) degree 4; positive leading coefficient; opens upward
   d) domain: $\{x \mid x \geq 0, x \in \mathrm{R}\}$; it is impossible to have negative skateboard sales
   e) The positive x-intercept represents the break-even point.
   f) $P(12) = 34\ 720$. The profit from the sale of 1200 skateboards is \$34 720.

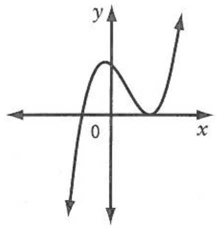
7. a) 3
   b) leading coefficient: 1; constant: 5; The constant represents the initial distance from the tree.
   c) degree 3; positive leading coefficient; extends from quadrant III to quadrant I
   d) domain: $\{t \mid t \geq 0, t \in \mathrm{R}\}$; time cannot be negative
   e) $D(7) = 12$; 12 m
   f)



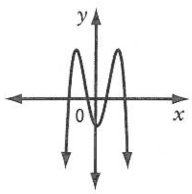
8. a) degree 4; positive leading coefficient; opens upward; extends from quadrant II to I; domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \geq 11\ 738, y \in \mathrm{R}\}$; The range for the time period $\{x \mid 0 \leq x \leq 15, x \in \mathrm{R}\}$ that the population model can be used is $\{y \mid 12\ 000 \leq y \leq 298\ 500, y \in \mathrm{R}\}$; no x-intercepts, y-intercept: 12 000
   b) 12 000
   c) 68 000
   d) approximately 13 years from now

9. Examples:

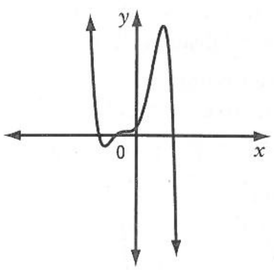
a)



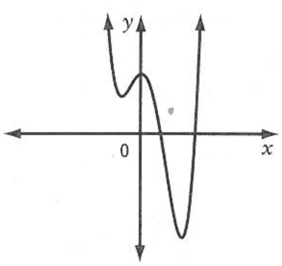
b)



c)



d)



3.2 The Remainder Theorem, pages 78–83

1. a) $\frac{x^3 + 3x^2 - 2x + 5}{x + 1} = x^2 + 2x - 4 + \frac{9}{x + 1}$
   b) $x \neq -1$
   c) $(x + 1)(x^2 + 2x - 4) + 9 = x^3 + 3x^2 - 2x + 5$
2. a) $Q(x) = 2x - 7$, $R = 26$
   b) $Q(x) = x^2 - 4x + 15$, $R = -70$
   c) $Q(x) = 3x^3 + 2x^2 - 6$, $R = 1$
   d) $Q(x) = -4x^3 - 12x^2 - 36x - 97$, $R = -298$
3. a) $\frac{2x^2 - x + 5}{x + 3} = 2x - 7 + \frac{26}{x + 3}$, $x \neq -3$
   b) $\frac{x^3 - x - 10}{x + 4} = x^2 - 4x + 15 - \frac{70}{x + 4}$, $x \neq -4$
   c) $\frac{3x^4 + 2x^3 - 6x + 1}{x} = 3x^3 + 2x - 6 + \frac{1}{x}$, $x \neq 0$
   d) $\frac{-4x^4 + 11x - 7}{x - 3} = -4x^3 - 12x^2 - 36x - 97 - \frac{298}{x - 3}$, $x \neq 3$
4. a) -36 b) 8
5. a) 22 b) -9 c) -2 d) -4
6. a) $k = 1$ b) $k = -1$ c) $k = -2$ d) $k = 2$
7. $m = 3$
8. $x - 2$, $3x + 1$
9. a) $(x - a)$ is a factor of $bx^3 + cx^2 + dx + e$.
   b) $e + ad + a^2c + a^3b$

3.3 The Factor Theorem, pages 84–90

1. a) $x - 2$ b) $x + 4$ c) $x - b$ d) $x + d$
2. a) Yes b) Yes c) No d) Yes
3. a) No b) Yes c) Yes d) Yes
4. a) $\pm1, \pm2, \pm4, \pm8, \pm16$
   b) $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$
   c) $\pm1, \pm2, \pm4, \pm8, \pm16, \pm32$
   d) $\pm1, \pm2, \pm3, \pm6, \pm9, \pm18$
5. a) $(x - 2)(x + 2)(x - 1)$
   b) $(x - 2)^2(x + 2)$
   c) $(x + 1)^3$
   d) $(x - 1)(x + 2)(x^2 + x + 1)$
6. a) $(x + 2)(x - 3)(x + 3)$
   b) $(2x + 1)(x - 1)(2x - 3)$
   c) $(x - 2)(2x + 5)(3x - 1)$
   d) $(x + 1)^2(x + 3)(x - 4)$
7. a) $k = 9$ b) $k = 3$
8. $x - 1$, $x + 3$, and $x + 5$
9. $V(x) = (x - 1)(x + 2)(3x - 1)$
10. Example: Start by using the integral zero theorem to check for a first possible integer value. Apply the factor theorem using the value found from the integral zero theorem. Use division to determine the remaining factor. Repeat the process until all factors are found.