3.2 The Remainder Theorem

KEY IDEAS

Long Division

You can use long division to divide a polynomial by a binomial: $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$

The components of long division are
- the dividend, $P(x)$, which is the polynomial that is being divided
- the divisor, $x - a$, which is the binomial that the polynomial is divided by
- the quotient, $Q(x)$, which is the expression that results from the division
- the remainder, R, which is the value or expression that is left over after dividing

To check the division of a polynomial, verify the statement $P(x) = (x - a)Q(x) + R$.

Synthetic Division
- a short form of division that uses only the coefficients of the terms
- it involves fewer calculations

Remainder Theorem
- When a polynomial $P(x)$ is divided by a binomial $x - a$, the remainder is $P(a)$.
- If the remainder is 0, then the binomial $x - a$ is a factor of $P(x)$.
- If the remainder is *not* 0, then the binomial $x - a$ is *not* a factor of $P(x)$.

Working Example 1: Divide a Polynomial by a Binomial of the Form $x - a$

a) Divide $P(x) = 9x + 4x^3 - 12$ by $x + 2$. Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

Solution

a) $x + 2\overline{)4x^3 + 0x^2 + 9x - 12}$

Why is the order of the terms different?
Why is it necessary to include the term $0x^2$?

See Example 1 on page 120 of *Pre-Calculus 12* for help with long division.

$\frac{4x^3 + 9x - 12}{x + 2} =$ ______________________

)) Since division by ________ is not defined, the divisor cannot be ________:

$x + 2 \neq$ ________ or $x \neq$ ________.

:) The corresponding statement that can be used to check the division is

________________________ = ________________________.

Working Example 2: Apply Polynomial Division to Solve a Problem

The volume, V, in cubic centimetres, of gift boxes is given by $V(x) = 2x^3 + x^2 - 27x - 36$. The height, h, in centimetres, is $x + 3$. What are the possible dimensions of the boxes in terms of x?

Solution

Divide the volume of the box by the height to obtain an expression for the area of the base of the box.

$x + 3\overline{)2x^3 + x^2 - 27x - 36}$

$\frac{V(x)}{h} = lw$, where lw is the area of the base

Since the remainder is ________, express the volume $2x^3 + x^2 - 27x - 36$ as (________)(________________).

The quotient ________________ represents the area of the base.

This expression can be factored as ________________________.

The factors represent the possible ________________ and ________________ of the base.

Expressions for the dimensions, in centimetres, are ________, ________, and ________.

Working Example 3: Divide a Polynomial Using Synthetic Division

a) Use synthetic division to divide $5x^2 - x + 2x^3 - 6$ by $x + 2$.

b) Check your results using long division.

Solution

a) Write the terms of the dividend in order of ________________ powers.
(*ascending* or *descending*)

____

Fill in the missing values and perform the division.

	______	______	______	______
	↓	↓	↓	↓
+2				
−				
×				

See Example 3 on page 122 of *Pre-Calculus 12* for help with synthetic division.

$(2x^3 + 5x^2 - x - 6) \div (x + 2) =$ ____________________; Restriction: ______________

b) $x + 2\overline{)2x^3 + 5x^2 - x - 6}$

The result obtained from long division is ______________ that using
(*the same as* or *different from*)
______________ division.

Working Example 4: Apply the Remainder Theorem

a) Use the remainder theorem to determine the remainder when $P(x) = 3x^4 - x^3 - 5$ is divided by $x - 3$.

b) Verify your answer using synthetic division.

Solution

a) Since the binomial is $x - 3$, determine the remainder by evaluating $P(x)$ at $x =$ ______,

or $P($______$)$.

$P($______$) = 3($______$)^4 - ($______$)^3 - 5$

$=$ ____________________

The remainder when $3x^4 - x^3 - 5$ is divided by $x - 3$ is ______.

Why is it necessary to write the polynomial this way?

b) To use synthetic division, first rewrite $P(x)$ as $P(x) =$ ______________.

	______	______	______	______	______
	↓	↓	↓	↓	↓
______					
−					
×					

The remainder when using synthetic division is ______.

Check Your Understanding

Practise

1. a) Use long division to divide $x^3 + 3x^2 - 2x + 5$ by $x + 1$. Express the result in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$.

   b) Identify any restrictions on the variable.

   c) Write the corresponding statement that can be used to check the division. Then, verify your answer.

2. Divide using long division. Then, verify your answer using synthetic division.

   a) $(2x^2 - x + 5) \div (x + 3)$

   b) $(x^3 - x - 10) \div (x + 4)$

c) $(3x^4 + 2x^3 - 6x + 1) \div x$

d) $(-4x^4 + 11x - 7) \div (x - 3)$

3. Express each result in #2 above in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.

Identify any restrictions on the variable.

a) $\frac{2x^2 - x + 5}{x + 3}$

b)

c)

d)

4. Determine the remainder when each polynomial function is divided by $x - 2$. Use the remainder theorem.

a) $P(x) = 2x^3 + 3x^2 - 17x - 30$

b) $P(x) = x^3 + x^2 - 4x + 4$

5. Determine each remainder.

a) $(6x^2 - x + 15) \div (x + 1)$

b) $(x^3 - x^2 - 2x - 1) \div (x + 2)$

c) $(2x^3 - 5x^2 - 13x + 2) \div (x - 4)$

d) $(x^4 - 3x^2 - 5x + 2) \div (x - 2)$

Apply

6. For each dividend, determine the value of k if the remainder is -2.

a) $(2x^3 - 5x^2 - 4x + k) \div (x + 1)$

b) $(x^3 - 4x^2 + kx + 10) \div (x - 3)$

c) $(3x^3 + kx^2 - 13x + 4) \div (x + 2)$

d) $(kx^3 - 4x^2 - 5x + 8) \div (x - 2)$

7. For what value of m will the polynomial $P(x) = x^3 + 6x^2 + mx - 4$ have the same remainder when it is divided by $x - 1$ and $x + 2$?

Since the remainder is the same, determine the value of m by solving $P(1) = P(\_\_\_\_\_\_\_\_)$.

8. You can model the volume, in cubic centimetres, of a rectangular box by the polynomial function $V(x) = 3x^3 + x^2 - 12x - 4$. Determine expressions for the other dimensions of the box if the height is $x + 2$.

Connect

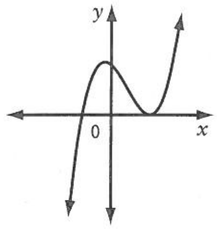
9. When the polynomial $bx^3 + cx^2 + dx + e$ is divided by $x - a$, the remainder is zero.

a) What can you conclude from this result?

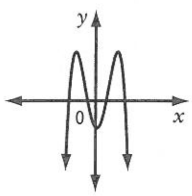
b) Write an expression for the remainder in terms of a, b, c, d, and e.

9. Examples:

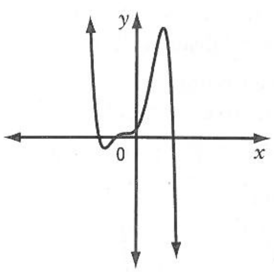
a)



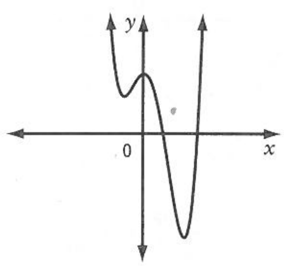
b)



c)



d)



3.2 The Remainder Theorem, pages 78–83

1. a) $\frac{x^3 + 3x^2 - 2x + 5}{x + 1} = x^2 + 2x - 4 + \frac{9}{x + 1}$

   b) $x \neq -1$

   c) $(x + 1)(x^2 + 2x - 4) + 9 = x^3 + 3x^2 - 2x + 5$

2. a) $Q(x) = 2x - 7, R = 26$

   b) $Q(x) = x^2 - 4x + 15, R = -70$

   c) $Q(x) = 3x^3 + 2x^2 - 6, R = 1$

   d) $Q(x) = -4x^3 - 12x^2 - 36x - 97, R = -298$

3. a) $\frac{2x^2 - x + 5}{x + 3} = 2x - 7 + \frac{26}{x + 3}, x \neq -3$

   b) $\frac{x^3 - x - 10}{x + 4} = x^2 - 4x + 15 - \frac{70}{x + 4}, x \neq -4$

   c) $\frac{3x^4 + 2x^3 - 6x + 1}{x} = 3x^3 + 2x - 6 + \frac{1}{x}, x \neq 0$

   d) $\frac{-4x^4 + 11x - 7}{x - 3} = -4x^3 - 12x^2 - 36x - 97 - \frac{298}{x - 3},$ $x \neq 3$

4. a) -36 b) 8

5. a) 22 b) -9 c) -2 d) -4

6. a) $k = 1$ b) $k = -1$ c) $k = -2$ d) $k = 2$

7. $m = 3$

8. $x - 2, 3x + 1$

9. a) $(x - a)$ is a factor of $bx^3 + cx^2 + dx + e$.

   b) $e + ad + a^2c + a^3b$

3.3 The Factor Theorem, pages 84–90

1. a) $x - 2$ b) $x + 4$ c) $x - b$ d) $x + d$

2. a) Yes b) Yes c) No d) Yes

3. a) No b) Yes c) Yes d) Yes

4. a) $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

   b) $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

   c) $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32$

   d) $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

5. a) $(x - 2)(x + 2)(x - 1)$

   b) $(x - 2)^2(x + 2)$

   c) $(x + 1)^3$

   d) $(x - 1)(x + 2)(x^2 + x + 1)$

6. a) $(x + 2)(x - 3)(x + 3)$

   b) $(2x + 1)(x - 1)(2x - 3)$

   c) $(x - 2)(2x + 5)(3x - 1)$

   d) $(x + 1)^2(x + 3)(x - 4)$

7. a) $k = 9$ b) $k = 3$

8. $x - 1, x + 3$, and $x + 5$

9. $V(x) = (x - 1)(x + 2)(3x - 1)$

10. Example: Start by using the integral zero theorem to check for a first possible integer value. Apply the factor theorem using the value found from the integral zero theorem. Use division to determine the remaining factor. Repeat the process until all factors are found.