3.3 The Factor Theorem

KEY IDEAS

Factor Theorem

The factor theorem states that $x - a$ is a factor of a polynomial $P(x)$ if and only if $P(a) = 0$.
If and only if means that the result works both ways. That is,

- if $x - a$ is a factor then, $P(a) = 0$
- if $P(a) = 0$, then $x - a$ is a factor of a polynomial $P(x)$

Integral Zero Theorem

- The integral zero theorem describes the relationship between the factors and the constant term of a polynomial. The theorem states that if $x - a$ is a factor of a polynomial $P(x)$ with integral coefficients, then a is a factor of the constant term of $P(x)$ and $x = a$ is an integral zero of $P(x)$.

Factor by Grouping

- If a polynomial $P(x)$ has an even number of terms, it may be possible to group two terms at a time and remove a common factor. If the binomial that results from common factoring is the same for each pair of terms, then $P(x)$ may be factored by grouping.

Steps for Factoring Polynomial Functions

To factor polynomial functions using the factor theorem and the integral zero theorem,

- use the integral zero theorem to list possible integer values for the zeros
- next, apply the factor theorem to determine one factor
- then, use division to determine the remaining factor
- repeat the above steps until all factors are found

Working Example 1: Use the Factor Theorem to Test for Factors of a Polynomial

Which binomials are factors of the polynomial $P(x) = x^3 + 4x^2 + x - 6$? Justify your answers.

a) $x - 1$ **b)** $x - 2$ **c)** $x + 2$ **d)** $x + 3$

Solution

Use the factor theorem to evaluate $P(a)$ given $x - a$.

a) For $x - 1$, substitute $x =$ ________ into the polynomial expression.

$P($________$) =$

Since the remainder is ________, $x - 1$ ______________ a factor of $P(x)$.
(*is* or *is not*)

b) For $x - 2$, substitute $x =$ ________ into the polynomial expression.

$P($________$) =$

Since the remainder is ________, $x - 2$ ________________ a factor of $P(x)$.
(*is* or *is not*)

c) For $x + 2$, substitute $x =$ ________ into the polynomial expression.

$P($________$) =$

Since the remainder is ________, $x + 2$ ________________ a factor of $P(x)$.
(*is* or *is not*)

d) For $x + 3$, substitute $x =$ ________ into the polynomial expression.

$P($________$) =$

Since the remainder is ________, $x + 3$ ________________ a factor of $P(x)$.
(*is* or *is not*)

Working Example 2: Factor Using the Integral Zero Theorem

a) Factor $2x^3 + 3x^2 - 3x - 2$ fully.

b) Describe how to use the factors of the polynomial expression to determine the zeros of the corresponding polynomial function.

Solution

a) Let $P(x) =$ ________________. Find a factor by evaluating $P(x)$ for values of x that are factors of ________________: ________________ and ________________.

Test the values until you find one that gives a remainder of zero.

$P($________$) =$

Since $P($________$) = 0$, ________ is a factor of $P(x)$.

Use synthetic or long division to find the other factors.

Which method of division do you prefer? Why?

Therefore, $2x^3 + 3x^2 - 3x - 2 = ($________________$)($________________$)($________________$)$.

b) Since the factors of $2x^3 + 3x^2 - 3x - 2$ are ______________, ______________, and ______________, the corresponding zeros of the function are ________, ________, and ________.

For an explanation of how the zeros can be confirmed, refer to Example 2b) on page 130 of *Pre-Calculus 12*.

Working Example 3: Factor Higher-Degree Polynomials

Fully factor $x^4 + 3x^3 - 7x^2 - 27x - 18$.

Solution

Let $P(x) =$ ______________.

Find a factor by testing factors of ______________: ______________________.
Test the values until you find one that gives a remainder of zero.

$P($________$) =$

Since $P($________$) = 0$, ______________ is a factor of $P(x)$.
Use division to find the other factors.

The remaining factor is ______________.

Let $f(x) =$ ______________________.
Use the factor theorem again.

Since $f($________$) = 0$, ______________ is a second factor.
Use division to find the other factors.

Combine all the factors to write the fully factored form.

Therefore, $x^4 + 3x^3 - 7x^2 - 27x - 18 =$ ______________________.

Compare this method with Method 2 of Example 3 on page 131 of *Pre-Calculus 12*. Is it possible to use factor by grouping in this situation? Explain.

Working Example 4: Solve Problems Involving Polynomial Expressions

An artist creates a carving from a block of soapstone. The soapstone is in the shape of a rectangular prism whose volume, in cubic feet, is represented by $V(x) = 6x^3 + 25x^2 + 2x - 8$, where x is a positive real number. What are the factors that represent possible dimensions, in terms of x, of the block of soapstone?

Solution

The possible integral factors correspond to the factors of the ______________ term of the polynomial, ______________: ______________________.

Use the factor theorem to determine which of these values correspond to the factors of the polynomial.

The values of x that result in a remainder of ________ are ______________, and the corresponding factors are ________________________.

The possible dimensions of the block of soapstone are ________________________.

See Example 4 on page 132 of *Pre-Calculus 12* for another method of solving this problem.

Check Your Understanding

Practise

1. What is the corresponding binomial factor of a polynomial, $P(x)$, given the value of the zero?

   a) $P(2) = 0$

   b) $P(-4) = 0$

   c) $P(b) = 0$

   d) $P(-d) = 0$

2. Determine whether $x + 1$ is a factor of each polynomial.

   a) $x^3 + x^2 - x - 1$

   b) $x^4 - 3x^3 - 4x^2 + x + 1$

   c) $2x^3 - x^2 - 3x - 1$

   d) $4x^4 + 7x + 3$

3. State whether each polynomial has $x + 3$ as a factor.

   a) $x^3 + x^2 - x + 6$

   b) $2x^3 + 9x^2 + 10x + 3$

   c) $x^3 + 27$

   d) $x^4 - 9x^2 + 2x + 6$

4. What are the possible integral zeros of each polynomial?

   a) $x^3 - 3x^2 + 4x - 16$

   b) $x^3 + 2x^2 + 8x + 12$

   c) $x^3 - 3x^2 + 10x - 32$

   d) $x^4 + 8x^3 - 9x^2 + 2x + 18$

5. Factor fully.

a) $x^3 - x^2 - 4x + 4$

b) $x^3 - 2x^2 - 4x + 8$

c) $x^3 + 3x^2 + 3x + 1$

d) $x^4 + 2x^3 - x - 2$

6. Factor fully.

a) $x^3 + 2x^2 - 9x - 18$

b) $4x^3 - 8x^2 + x + 3$

c) $6x^3 + x^2 - 31x + 10$

d) $x^4 + x^3 - 13x^2 - 25x - 12$

Apply

7. Determine the value(s) of k so that the binomial is a factor of the polynomial.

   a) $P(x) = x^3 + 5x^2 + kx + 6$ $\qquad x + 2$

   If $x + 2$ is a factor, then $P(\_\_\_\_\_\_) = \_\_\_\_\_\_$.

   b) $P(x) = kx^3 - 10x^2 + 2x + 3$ $\qquad x - 3$

8. The product of four integers is $x^4 + 7x^3 + 7x^2 - 15x$, where x is one of the integers. What are the possible expressions for the other three integers?

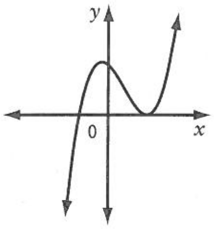
9. A sculptor creates a carving from a block of marble. The marble is in the shape of a rectangular prism whose volume, in cubic feet, is represented by $V(x) = 3x^3 + 2x^2 - 7x + 2$, where x is a positive real number. What are the factors that represent possible dimensions, in terms of x, of the block of marble?

Connect

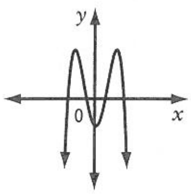
10. Describe the steps required to factor the polynomial $x^4 - 10x^3 + 24x^2 + 10x - 25$.

9. Examples:

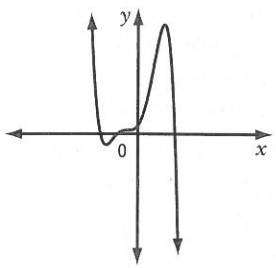
a)



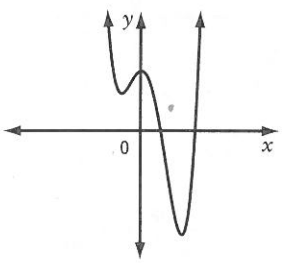
b)



c)



d)



3.2 The Remainder Theorem, pages 78–83

1. a) $\frac{x^3 + 3x^2 - 2x + 5}{x + 1} = x^2 + 2x - 4 + \frac{9}{x + 1}$
   b) $x \neq -1$
   c) $(x + 1)(x^2 + 2x - 4) + 9 = x^3 + 3x^2 - 2x + 5$
2. a) $Q(x) = 2x - 7$, $R = 26$
   b) $Q(x) = x^2 - 4x + 15$, $R = -70$
   c) $Q(x) = 3x^3 + 2x^2 - 6$, $R = 1$
   d) $Q(x) = -4x^3 - 12x^2 - 36x - 97$, $R = -298$
3. a) $\frac{2x^2 - x + 5}{x + 3} = 2x - 7 + \frac{26}{x + 3}$, $x \neq -3$
   b) $\frac{x^3 - x - 10}{x + 4} = x^2 - 4x + 15 - \frac{70}{x + 4}$, $x \neq -4$
   c) $\frac{3x^4 + 2x^3 - 6x + 1}{x} = 3x^3 + 2x - 6 + \frac{1}{x}$, $x \neq 0$
   d) $\frac{-4x^4 + 11x - 7}{x - 3} = -4x^3 - 12x^2 - 36x - 97 - \frac{298}{x - 3}$, $x \neq 3$
4. a) -36 b) 8
5. a) 22 b) -9 c) -2 d) -4
6. a) $k = 1$ b) $k = -1$ c) $k = -2$ d) $k = 2$
7. $m = 3$
8. $x - 2$, $3x + 1$
9. a) $(x - a)$ is a factor of $bx^3 + cx^2 + dx + e$.
   b) $e + ad + a^2c + a^3b$

3.3 The Factor Theorem, pages 84–90

1. a) $x - 2$ b) $x + 4$ c) $x - b$ d) $x + d$
2. a) Yes b) Yes c) No d) Yes
3. a) No b) Yes c) Yes d) Yes
4. a) $\pm1, \pm2, \pm4, \pm8, \pm16$
   b) $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$
   c) $\pm1, \pm2, \pm4, \pm8, \pm16, \pm32$
   d) $\pm1, \pm2, \pm3, \pm6, \pm9, \pm18$
5. a) $(x - 2)(x + 2)(x - 1)$
   b) $(x - 2)^2(x + 2)$
   c) $(x + 1)^3$
   d) $(x - 1)(x + 2)(x^2 + x + 1)$
6. a) $(x + 2)(x - 3)(x + 3)$
   b) $(2x + 1)(x - 1)(2x - 3)$
   c) $(x - 2)(2x + 5)(3x - 1)$
   d) $(x + 1)^2(x + 3)(x - 4)$
7. a) $k = 9$ b) $k = 3$
8. $x - 1$, $x + 3$, and $x + 5$
9. $V(x) = (x - 1)(x + 2)(3x - 1)$
10. Example: Start by using the integral zero theorem to check for a first possible integer value. Apply the factor theorem using the value found from the integral zero theorem. Use division to determine the remaining factor. Repeat the process until all factors are found.