3.4 Equations and Graphs of Polynomial Functions

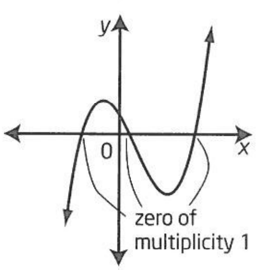
KEY IDEAS

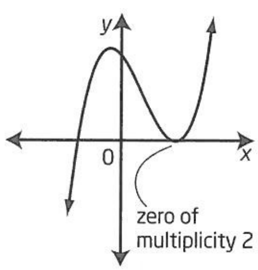
Sketching Graphs of Polynomial Functions

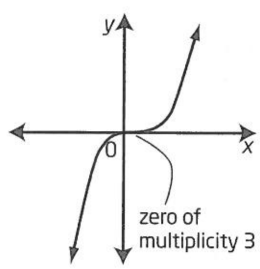
- To sketch the graph of a polynomial function, use the x-intercepts, the y-intercept, the degree of the function, and the sign of the leading coefficient.
- The x-intercepts of the graph of a polynomial function are the roots of the corresponding polynomial equation.
- Determine the zeros of a polynomial function from the factors.
- Use the factor theorem to express a polynomial function in factored form.

Multiplicity of a Zero

- If a polynomial has a factor $x - a$ that is repeated n times, then $x = a$ is a zero of multiplicity n.
- The multiplicity of a zero or root can also be referred to as the *order* of the zero or root.
- The shape of a graph of a polynomial function close to a zero of $x = a$ (multiplicity n) is similar to the shape of the graph of a function with degree equal to n of the form $y = (x - a)^n$.
- Polynomial functions change sign at x-intercepts that correspond to *odd* multiplicity. The graph crosses over the x-axis at these intercepts.
- Polynomial functions do not change sign at x-intercepts of *even* multiplicity. The graph touches, but does not cross, the x-axis at these intercepts.







Transformation of Polynomial Functions

To sketch the graph of a polynomial function of the form $y = a[b(x - h)]^n + k$ or $y - k = a[b(x - h)]^n$, where $n \in \mathrm{N}$, apply the following transformations to the graph of $y = x^n$.

Note: You may apply the transformations represented by a and b in any order before the transformations represented by h and k.

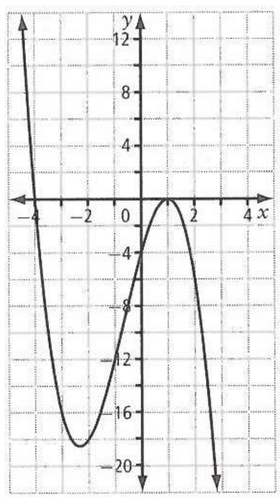
Parameter	Transformation
k	• Vertical translation up or down • $(x, y) \rightarrow (x, y + k)$
h	• Horizontal translation left or right • $(x, y) \rightarrow (x + h, y)$
a	• Vertical stretch about the x-axis by a factor of $\lvert a \rvert$ • For $a < 0$, the graph is also reflected in the x-axis • $(x, y) \rightarrow (x, ay)$

b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$ • For $b < 0$, the graph is also reflected in the y-axis • $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$

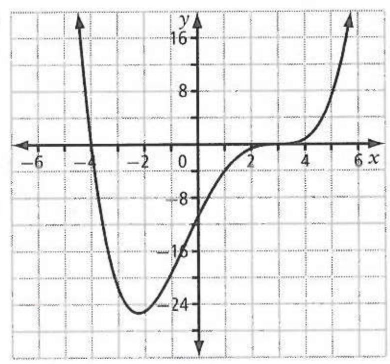
Working Example 1: Analyse Graphs of Polynomial Functions

For each graph of a polynomial function, determine
- the least possible degree
- the sign of the leading coefficient
- the x-intercepts and the factors of the function with least possible degree
- the intervals where the function is positive and the intervals where it is negative

a)



b)



Solution

a) There are ________ x-intercepts; they are ____________________________.

The x-intercept of multiplicity 1 is ________.

The x-intercept of multiplicity 2 is ________.

The least possible degree of the graph is ________.

The graph extends from quadrant ________ to quadrant ________.

The leading coefficient is ________________.
(*positive* or *negative*)

The factors are ________________________________.

The function is positive for values of x in the interval(s) ____________________________.

The function is negative for values of x in the interval(s) ____________________________.

b) The x-intercepts are ______________________.

The x-intercept of multiplicity ________ is ________.

The x-intercept of multiplicity ________ is ________.

The least possible degree of the graph is ________.

The graph extends from quadrant ________ to quadrant ________.

The leading coefficient is ________________.
(*positive* or *negative*)

The factors are ______________________.

The function is positive for values of x in the interval(s) ______________________.

The function is negative for values of x in the interval(s) ______________________.

To see additional graphs, refer to Example 1 on page 138 of *Pre-Calculus 12*.

Working Example 2: Analyse Equations to Sketch Graphs of Polynomial Functions

Sketch the graph of each polynomial function.

a) $y = -(x + 1)^3(x - 3)$

b) $y = 2x^5 + x^4 - 18x^3 - 9x^2$

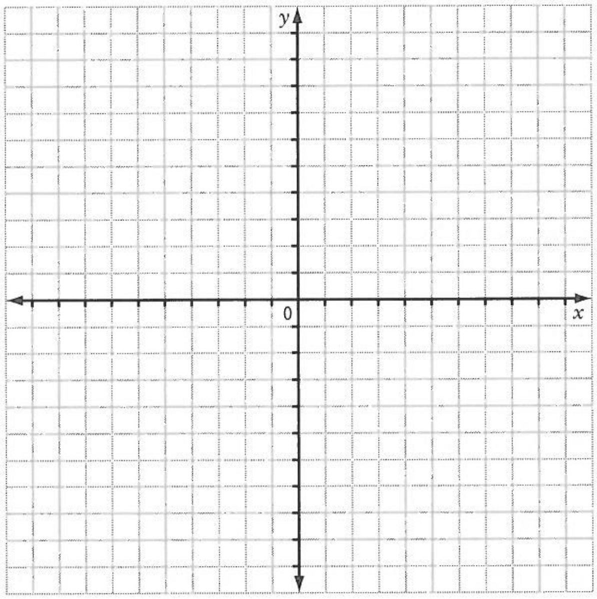
Solution

a) The function $y = -(x + 1)^3(x - 3)$ is in ________________ form.

Degree	
Leading coefficient	
End behaviour	
Zeros/x-intercepts	
Multiplicity of zeros	
y-intercept	
Interval(s) where the function is positive	
Interval(s) where the function is negative	

Use the information from the table to sketch the graph.

How can you check whether the function is positive or negative?



b) The function $y = 2x^5 + x^4 - 18x^3 - 9x^2$ is not in

_______________ form.

First, factor out the common factor.

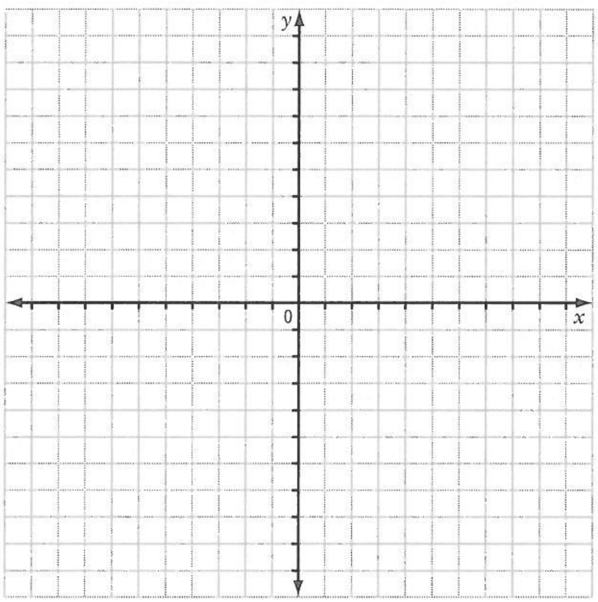
What zero corresponds to the common factor? What multiplicity does it have?

Next, use the integral zero theorem and the factor theorem to factor the polynomial.

The factored form of $y = 2x^5 + x^4 - 18x^3 - 9x^2$ is ___________________________.

Degree	
Leading coefficient	
End behaviour	
Zeros/x-intercepts	
Multiplicity of zeros	
y-intercept	
Interval(s) where the function is positive	
Interval(s) where the function is negative	

Use the information from the table to sketch the graph.



Working Example 3: Apply Transformations to Sketch a Graph

The graph of $y = x^4$ is transformed to obtain the graph of $y = 3\left[-\frac{1}{2}(x + 1)\right]^4 - 4$.

a) State the parameters and describe the corresponding transformations.

b) Complete a table to show what happens to the given points under each transformation.

c) Sketch the graph of $y = 3\left[-\frac{1}{2}(x + 1)\right]^4 - 4$.

Solution

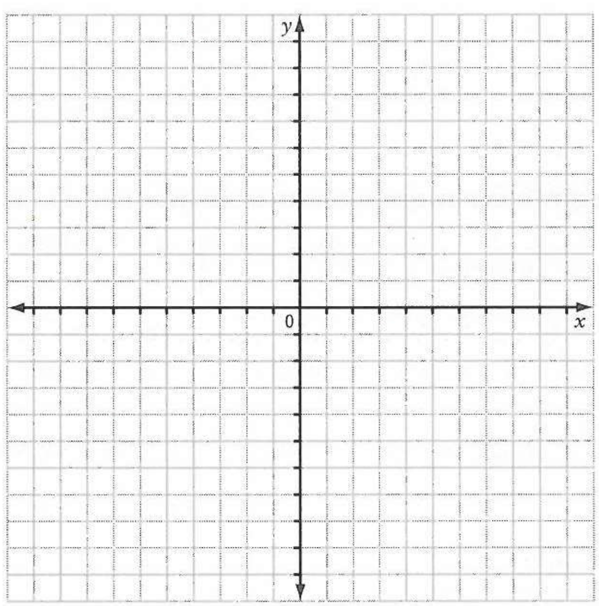
a)

Parameter	Description of Transformation
$b =$	
$a =$	
$h =$	
$k =$	

b) Complete the table to show what happens to the given points under each transformation.

$y = x^4$	$y = \left(-\frac{1}{2}x\right)^4$	$y = 3\left(-\frac{1}{2}x\right)^4$	$y = 3\left[-\frac{1}{2}(x + 1)\right]^4 - 4$
(−2, 16)			
(−1, 1)			
(0, 0)			
(1, 1)			
(2, 16)			

c) To sketch the graph, plot the points from column 4 and draw a smooth curve through them.



Check Your Understanding

Practise

1. Solve.

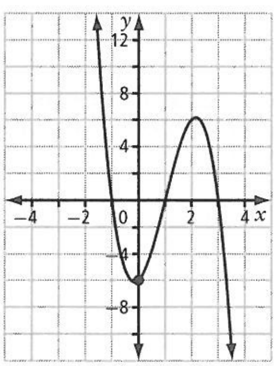
   a) $4x^3(x + 2)(2x - 1) = 0$

   b) $(x + 1)^2(x - 3)(x - 5) = 0$

   c) $x^3 - 8 = 0$

2. Use the graph of the given function to write the corresponding polynomial equation. State the roots of the equation. The roots are all integral values.

   a)



The graph of the function has ________ x-intercepts.

It crosses the x-axis at each of the x-intercepts. All the x-intercepts are of ________________ multiplicity.
(*even* or *odd*)

The least possible multiplicity of each x-intercept is ________, so the least possible degree is ________.

The graph extends up into quadrant ________ and down into quadrant ________, so the leading coefficient is ________________.
(*positive* or *negative*)

The y-intercept is ________; this is the ________________ term in the equation of the function.

The zeros, or x-intercepts, are ________, ________, and ________. The product of the roots is ________________.

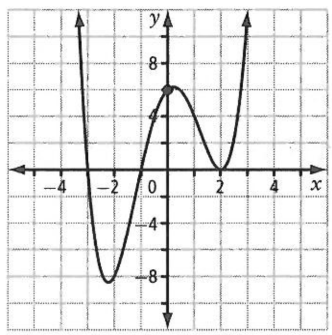
Compare the product of the roots to the y-intercept to determine the vertical stretch, a.

$a =$ ________

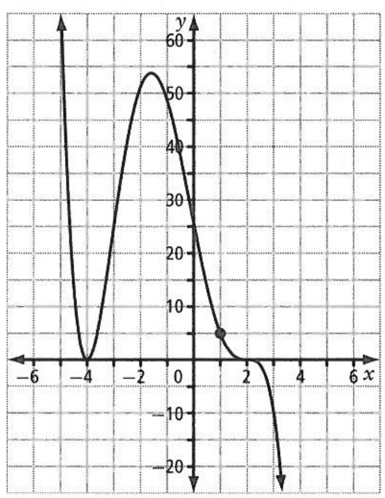
The equation of the polynomial function is

$f(x) =$ ________(________)(________)(________).

b)



c)

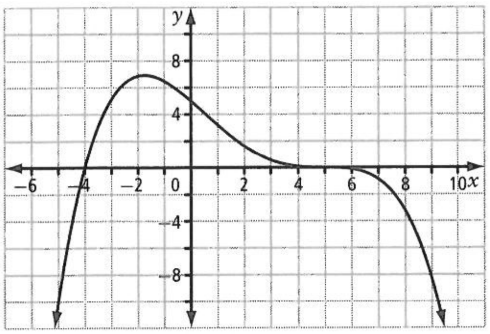


In this case, use the coordinates of the point (1, 5) to solve for *a*.

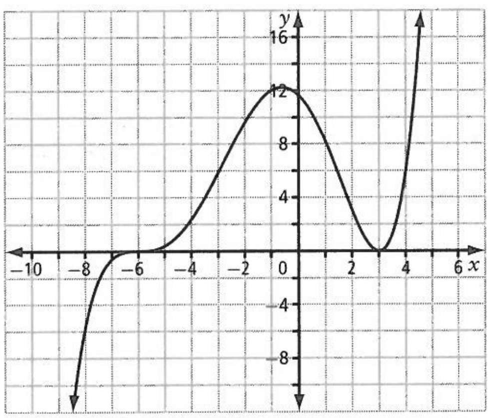
3. For each graph,

- state the x-intercepts
- state the intervals where the function is positive and the intervals where the function is negative
- explain whether the graph might represent a polynomial that has zero(s) of multiplicity 1, 2, or 3.

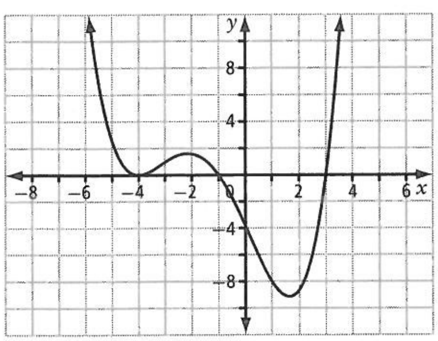
a)



b)

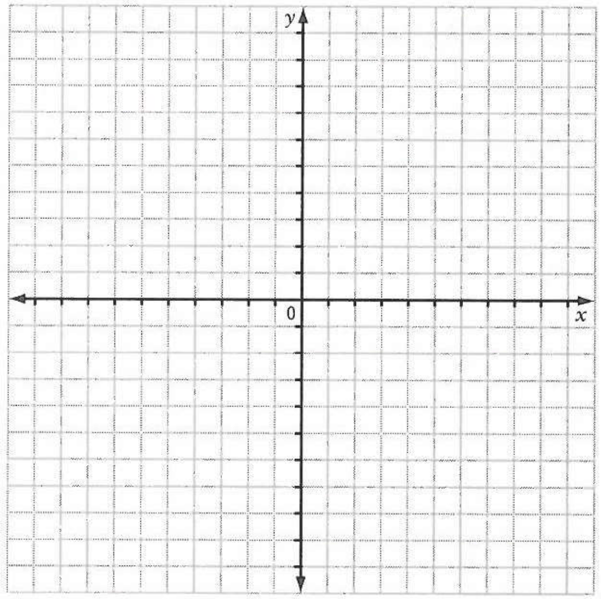


c)

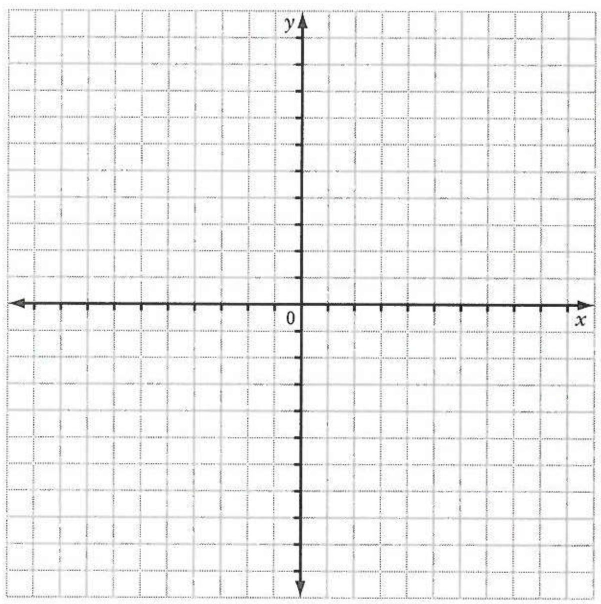


4. Without using technology, sketch the graph of each function. Label all intercepts. (Hint: Factor.)

a) $f(x) = -2x^3 + 3x^2 + 11x - 6$



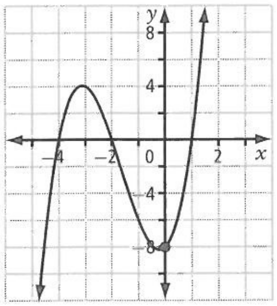
b) $g(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$



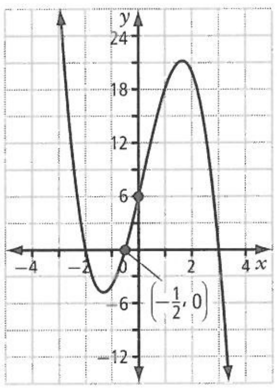
Apply

5. Determine the equation for each graph of a polynomial function shown.

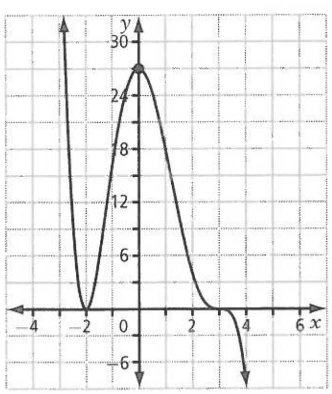
a)



b)



c)



6. **a)** Given the function $y = x^3$, list the parameters of the transformed polynomial function $y = \frac{1}{2}\left(3(x + 4)\right)^3 - 5$ and describe how each parameter transforms the graph of the function $y = x^3$.

b) Determine the domain and range for the transformed function.

7. Determine the equation with least degree for each polynomial function.

a) quartic function with zeros 2 (multiplicity 3) and -5, and y-intercept 30

b) quintic function with zeros -1 (multiplicity 2), 3 (multiplicity 1), and -2 (multiplicity 2), and constant term -12

8. An interlocking stone path that is x feet wide is built around a rectangular garden. The garden is 20 ft wide and 40 ft long. The combined surface area of the garden and the walking path is 1196 ft^2. What are the dimensions of the stone path?

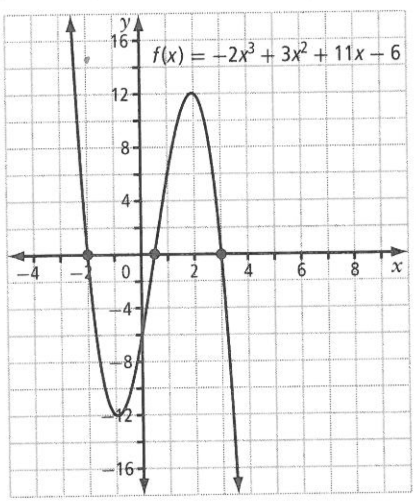
For help with #8, see Example 4 on page 145 of *Pre-Calculus 12* for an example of how to solve a problem involving polynomial functions.

Connect

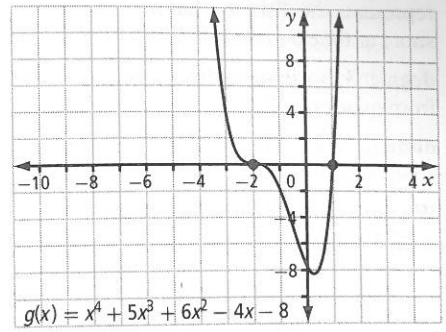
9. Given a polynomial function of the form $y = a[b(x - h)]^n + k$, which parameters do not change the shape of the graph of the function? Explain.

3.4 Equations and Graphs of Polynomials Functions, pages 91–102

1. a) $x = 0, -2, \frac{1}{2}$
   b) $x = -1, 3, 5$
   c) $x = 2$
2. a) $f(x) = -2(x - 1)(x + 1)(x - 3)$; −1, 1, 3
   b) $f(x) = 0.5(x - 2)^2(x + 1)(x + 3)$; −1, −3, 2
   c) $f(x) = -0.2(x - 2)^3(x + 4)^2$; −4, 2
3. a) −4 and 5; positive for $-4 < x < 5$; negative for $x < -4$ and $x > 5$; −4 (multiplicity 1) and 5 (multiplicity 3); the function changes sign at both, but is flatter at $x = 5$
   b) −6 and 3; positive for $-6 < x < 3$ and $x > 3$; negative for $x < -6$; −6 (multiplicity 3) and 3 (multiplicity 2); the function changes sign at $x = -6$, but not at $x = 3$
   c) −4, −1, and 3; positive for $x < -4$, $-4 < x < -1$, and $x > 3$; negative for $-1 < x < 3$; −4 (multiplicity 2), −1 (multiplicity 1), and 3 (multiplicity 1); the function changes sign at $x = -1$ and at $x = 3$, but not at $x = -4$
4. a) x-intercepts: −2, 0.5, 3 (all of multiplicity 1); y-intercept: −6



   b) x-intercepts: −2 (multiplicity 3) and 1 (multiplicity 1); y-intercept: −8



5. a) $f(x) = (x + 4)(x - 1)(x + 2)$
   b) $f(x) = -(2x + 1)(x - 3)(x + 2)$
   c) $f(x) = -0.25\,(x + 2)^2(x - 3)^3$
6. a) $a = \frac{1}{2}$; vertical stretch by a factor of $\frac{1}{2}$
   $b = 3$; horizontal stretch by a factor of $\frac{1}{3}$
   $h = -4$; translation of 4 units left
   $k = -5$; translation of 5 units down
   b) domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \in \mathrm{R}\}$
7. a) $y = -\frac{3}{4}(x - 2)^3(x + 5)$
   b) $y = (x + 1)^2(x - 3)(x + 2)^2$
8. 26 ft by 46 ft
9. h and k; these parameters represent the horizontal translation and the vertical translation, respectively, of the graph and do not change its shape or orientation.

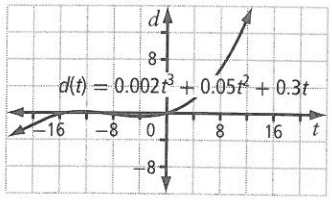
Chapter 3 Review, pages 103–107

1.

Polynomial Function	Degree	Type	Leading Coefficient	Constant Term
a) $f(x) = -2x^4 - x^3 + 3x - 7$	4	Quartic	−2	−7
b) $y = 3x^5 + 2x^4 - x^3 + 3$	5	Quintic	3	3
c) $g(x) = 0.5x^3 - 8x^2$	3	Cubic	0.5	0
d) $p(x) = 10$	0	Constant	0	10

2. a) even degree; negative leading coefficient; 2 x-intercepts; domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \leq 19, y \in \mathrm{R}\}$
   b) odd degree; positive leading coefficient; 3 x-intercepts; domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \in \mathrm{R}\}$

3. a) degree 3
   b) leading coefficient: 0.002; constant: 0; The constant represents the distance that the boat is from the shore at time 0 s (the initial position of the boat).
   c) degree: 3; positive leading coefficient; extends from quadrant III to I
   d) domain: $\{t \mid t \geq 0, t \in R\}$; it is impossible to have negative time
   e) When $t = 15$, $d(15) = 22.5$. After 15 s, the boat is 22.5 m from the shore.
   f)

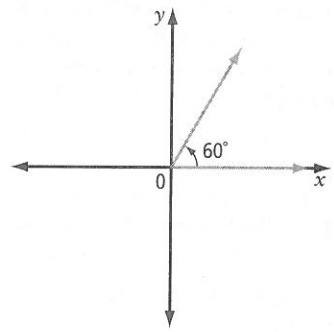


4. a) $\frac{5x^3 - 7x^2 - x + 6}{x - 1} = 5x^2 - 2x - 3 + \frac{3}{x - 1}$
   b) $x \neq 1$
   c) $(x - 1)(5x^2 - 2x - 3) + 3 = 5x^3 - 7x^2 - x + 6$
5. a) $R = 9$ b) $R = 15$
   c) $R = 41$ d) $R = 595$
6. a) $m = 4$ b) 28
7. $P(x) = x^3 + 2x^2 - 15x + 10$
8. a) $x - 7$ b) $x + 6$ c) $x - c$
9. a) Yes b) No
10. a) $\pm1, \pm3, \pm9, \pm27$
    b) $\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm36$
11. a) $(x - 3)(x - 2)(x + 1)$
    b) $(x - 4)(x + 2)(3x + 1)$
    c) $(x - 3)(x - 1)(x + 6)(5x + 2)$
    d) $x(x - 2)(x + 2)(2x + 5)$
12. $x - 1$, $x + 2$, and $5x + 2$
13. a) degree 5; negative leading coefficient; −3 (multiplicity 2) and 1 (multiplicity 3); the function changes sign at $x = 1$, but not at $x = -3$; positive for $x < -3$ and $-3 < x < 1$; negative for $x > 1$; $f(x) = -0.25(x + 3)^2(x - 1)^3$
    b) degree 4; positive leading coefficient; −2 (multiplicity 1), −0.5 (multiplicity 1), and 2 (multiplicity 2); the function changes sign at $x = -2$ and at $x = -0.5$, but not at $x = 2$; positive for $x < -2$, $-0.5 < x < 2$, and $x > 2$; negative for $-2 < x < -0.5$; $f(x) = 0.5(x + 2)(2x + 1)(x - 2)^2$
14. a) $a = -2$; vertical stretch by a factor of 2 and reflection in the x-axis
    $b = \frac{1}{3}$; horizontal stretch by a factor of 3
    $h = 1$; translation of 1 unit to the right
    $k = 4$; translation of 4 units up
    b) domain: $\{x \mid x \in R\}$; range: $\{y \mid y \in R\}$
15. $y = -3(x + 2)^2(x - 3)$

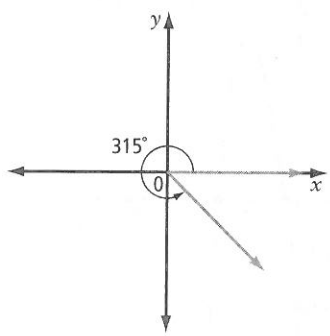
Chapter 4

4.1 Angles and Angle Measure, pages 109–119

1. a) $\frac{\pi}{3}$



   b) $\frac{7\pi}{4}$



   c) $-\frac{7\pi}{6}$